Lecture 2

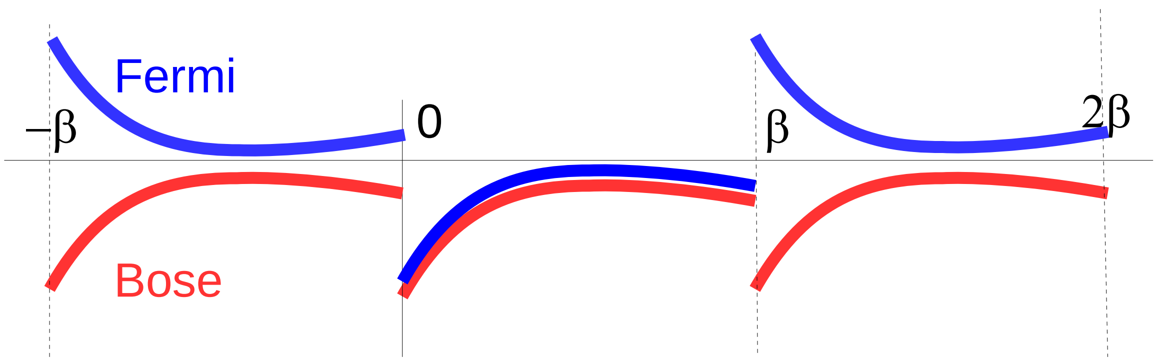
Green's functions

The Green's functions in Matsubara domain are defined as follows

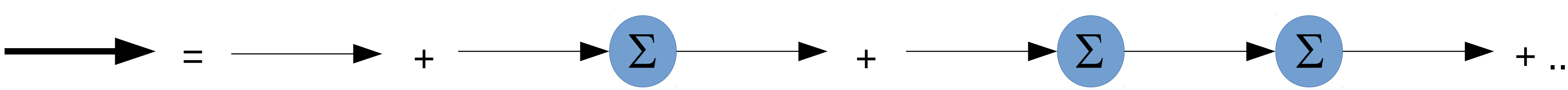
$$G^B_{\bar{1}1} = - < T\hat{a}_{\bar{1}}\hat{a}_1^\dagger > \qquad G^F_{\bar{1}1} = - < T\hat{c}_{\bar{1}}\hat{c}_1^\dagger >$$

Commuting/anticommuting property is included in T, so that

$$G^{B/F}_\tau = \pm G_{\tau+\beta}, \qquad G_{\bar{\tau}-\tau} = G_{\bar{\tau},\tau}.$$



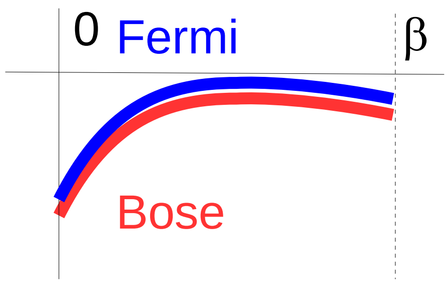
Let $\hat{H} = \hat{H}_0 + \hat{W}$, and $\hat{H}_0$ corresponds to the Green's function G_0.
The self-energy is $\Sigma = G_0^{-1} - G^{-1}$.

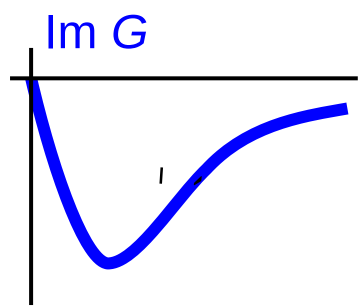


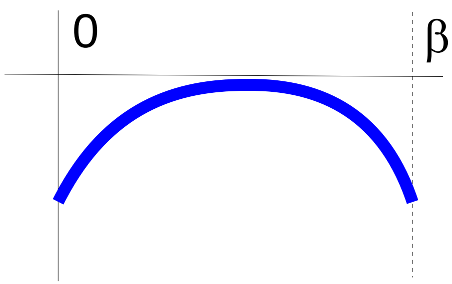
Examples:

$$\hat{H} = \varepsilon \hat{a}^\dagger \hat{a} \;\Rightarrow\; G_\tau = -\frac{e^{-\varepsilon\tau}}{1 - e^{-\varepsilon\beta}} \quad G_\omega = \frac{1}{i\omega - \varepsilon}$$

$$\hat{H} = \varepsilon \hat{c}^\dagger \hat{c} \;\Rightarrow\; G_\tau = -\frac{e^{-\varepsilon\tau}}{1 + e^{-\varepsilon\beta}} \quad G_\nu = \frac{1}{i\nu - \varepsilon}$$







$$\hat{H} = U\left(\hat{c}_\uparrow^\dagger \hat{c}_\uparrow - \frac{1}{2}\right)\left(\hat{c}_\downarrow^\dagger \hat{c}_\downarrow - \frac{1}{2}\right), \; \beta \gg U^{-1} \;\Rightarrow$$

$$G_\tau = -\frac{e^{-U\tau/2} + e^{-U(\beta-\tau)/2}}{2(1 + e^{-\beta U/2})}, \quad G_\nu = \frac{1/2}{i\nu - U/2} + \frac{1/2}{i\nu + U/2}, \quad \Sigma_\nu = \frac{U^2}{4}\frac{1}{i\nu}$$

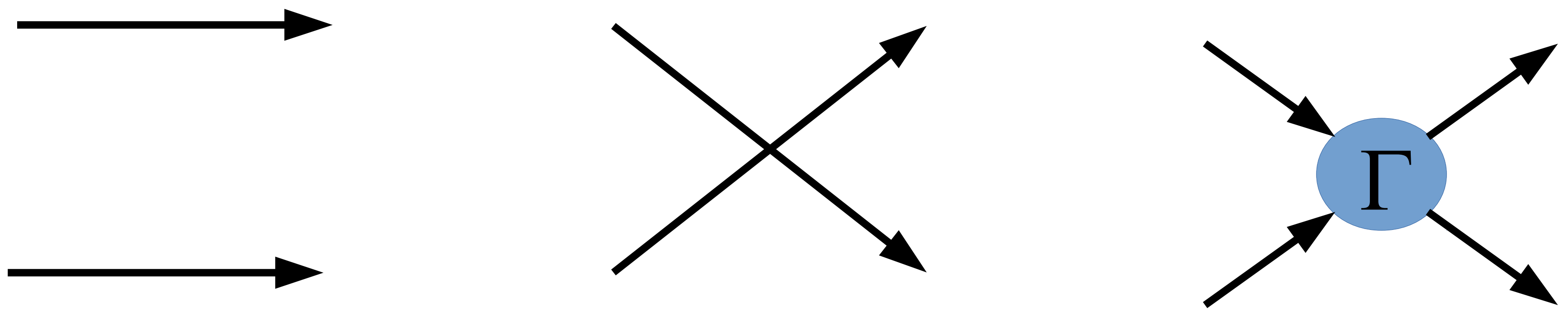
Lecture 2. *Green's functions*
Chapter 1. *Definitions*

Two-particle Green's function:

$$G^{(2)}_{\bar{1}\bar{2}21} =< T\hat{a}_{\bar{1}}\hat{a}_{\bar{2}}\hat{a}^{\dagger}_{2}\hat{a}^{\dagger}_{1} >$$

Vertex function is defined from the formula

$$G^{(2)}_{\bar{1}\bar{2}21} = G_{\bar{1}1}G_{\bar{2}2} \pm G_{\bar{1}2}G_{\bar{2}1} + G_{\bar{1}1'}G_{\bar{2}2'}\Gamma_{1'2'\bar{2}'\bar{1}'}G_{\bar{2}'2}G_{\bar{1}'1}.$$



Lecture 2. *Green's functions*

Chapter 1. *Definitions*

Recall

$$Z=\int e^{(\bar a {G_0}^{-1} a)-W[\bar a,a]} Da,\bar a$$

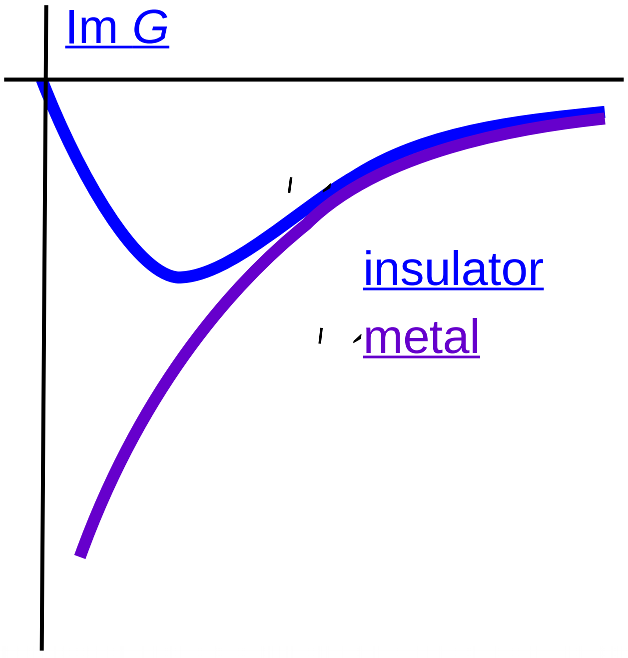
$\bar a_{\bar 1}(G_0)^{-1}_{\bar 1 1}a_1=\int \bar a_{\bar 1}(\partial_\tau-\epsilon_{1\bar 1})a_1 d\tau.$

In the path-integral formalizm, $\hat a_\tau,\hat a^\dagger_\tau\to\bar a_\tau,a_\tau$, therefore

$$G_{B/F}=\mp\frac{\partial}{\partial {G_0}^{-1}}\ln Z$$

For a Gaussian system $W=0$

$$G=G_0=\frac{1}{i\nu-\varepsilon_k}$$



Consider new variables $a \to a + \alpha$ in the path-integral:

$$Z = \int e^{-S[a+\alpha, \bar{a}+\bar{\alpha}]} Da, \bar{a}.$$

Since $Z(\alpha) = const$, and therefore $\frac{\partial \ln Z}{\partial \alpha} = 0$,

$$\left\langle \frac{\partial S}{\partial \bar{a}} \right\rangle = 0, \quad \left\langle \frac{\partial S}{\partial a} \right\rangle = 0.$$

Similarly, $a \to a + \alpha \cdot a \quad \Rightarrow \quad \frac{\partial \ln Z}{\partial \alpha_{\bar{1}1}} = \delta_{\bar{1}1} \quad \Rightarrow$

$$\left\langle \frac{\partial S}{\partial a_{\bar{1}}} a_1 \right\rangle = \delta_{\bar{1}1}.$$

Assuming Gaussian statistics allows to find $\langle a \rangle$ and $G = -\langle \bar{a} \cdot a \rangle$ from the two equations.

Next-order equations arise from $a \to a + \alpha \cdot a \cdot a$, ... , and allow (in principle) to find out higher momenta.

Let's introduce the Gaussian part of the action

$$Z = \int e^{(\bar{a}+\bar{\alpha})G_0{}^{-1}(a+\alpha)-W[a_\tau+\alpha_\tau,\bar{a}_\tau+\bar{\alpha}_\tau]} Da, \bar{a}.$$

Schwinger-Dyson equations read

$$< a >= G_0 < \partial_{\bar{a}} W >,$$

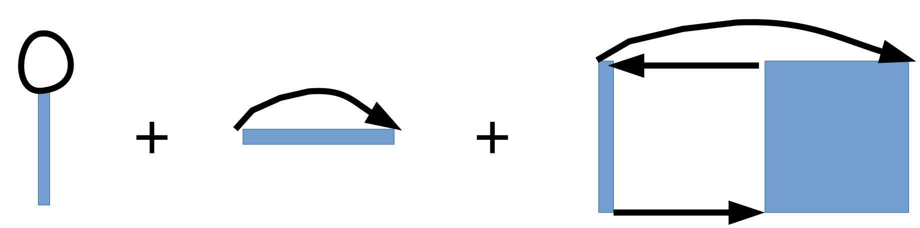
and, assuming $\langle a \rangle = 0$,

$$\Sigma = - < \bar{a} \cdot G^{-1} \partial_{\bar{a}} W > .$$

For $\hat{W} = \int \frac{1}{2} \hat{a}_1^\dagger \hat{a}_1 V_{1-2} \hat{a}_2^\dagger \hat{a}_2 d\tau$ this formula gives the Dyson equation :

$$\Sigma_P = V_{P=0} n - V_{P-Q} G_P - \Gamma_{Q'+P'=Q+P} V_{P-Q} G_{P'} G_{Q'} G_Q$$

where $P = (\omega, p)$.
This expression holds also for fermions.



For systems with density-density interaction, the gauge transformation

$$a_R \to a_R(1 + i0\phi_R)$$

does not vary W for any $\phi_{R=(r,t)}$ (expressions are valid in (r,t) space only!). We get

$$\sum_{RR'} (G_0)^{-1}_{RR'} G_{R'R} \phi_R = \sum_{RR'} (G_0)^{-1}_{RR'} G_{R'R} \phi_{R'}.$$

R.h.s. equals $\sum_{RR'} (G_0)^{-1}_{R'R} G_{RR'} \phi_R$ (summation index change).
The conservation law reads $\sum_{R'} (G_0)^{-1}_{RR'} G_{R'R} = \sum_{R'} (G_0)^{-1}_{R'R} G_{RR'}$ for any R, or shortly (diagonal matrix elements in (r,t) space are vanished):

$$\left[G_0^{-1}, G\right]_{RR} = 0$$

In real-time, for $\hat{H}_0 = \varepsilon_{rr'} \hat{a}^\dagger_r \hat{a}_{r'}$, the variance of action is $\int \left(\bar{a}_{rt} a_{rt} \partial_t \phi_{rt} + i\varepsilon_{r,r'} \bar{a}_{rt} a_{r't} (\phi_{rt} - \phi_{r't})\right) dt$. Integration by parts gives

$$\partial_t n_r = \sum_{r'} \left(\varepsilon_{rr'} G_{r'r,t'=t+0} - G_{rr',t'=t+0} \varepsilon_{r'r}\right) \equiv \left[\varepsilon, G_{t'=t+0}\right]_{rr}.$$

Let's vary the conservation law:

$$\frac{\delta}{\delta {G_0}^{-1}} \sum_{R'R} (G_0)^{-1}_{RR'} G_{R'R}(\phi_{R'} - \phi_R) = 0.$$

Calculation gives

$$G_{R'R}(\phi_{R'} - \phi_R) = \sum_{R_2 R_2'} (G_0)^{-1}_{R_2 R_2'} (G^{(2)}_{R' R_2' R_2 R} - G_{R_2' R_2} G_{R'R})(\phi_{R_2'} - \phi_{R_2})$$

Multiply the equation by $\phi^{-1}_{R'}$ and take $\phi_R = e^{iKR}$:

$$\begin{aligned} & G_{R'R} - G_{R'R} e^{iK(R-R')} = \\ & = \sum_{R_2 R_2'} (G_0)^{-1}_{R_2 R_2'} (G^{(2)}_{R' R_2' R_2 R} - G_{R_2' R_2} G_{R'R}) \left(e^{iK(R_2' - R')} - e^{iK(R_2 - R')} \right). \end{aligned}$$

Perform the Fourier transform and divide the equation by $G_P G_{P+K}$:

$$G^{-1}_{P+K} - G^{-1}_P = \sum_Q \left(\Gamma_{P,Q+K,Q,P+K} G_Q G_{Q+K} - \delta_{PQ} \right) \left((G_0)^{-1}_Q - (G_0)^{-1}_{Q+K} \right).$$

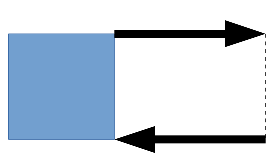
We arrive to

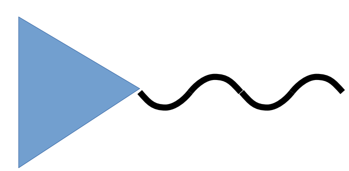
$$\Sigma_P - \Sigma_{P+K} = \sum_Q \Gamma_{P,Q+K,Q,P+K} G_Q G_{Q+K} \left((G_0)^{-1}_Q - (G_0)^{-1}_{Q+K} \right).$$

The equation holds for any P, K. Let's take the limit $K \to 0 \;\Rightarrow$.

$$\partial_P \Sigma_P = \sum_Q \Gamma_{P,Q,Q,P} G_Q^2 \partial_Q (G_0)^{-1}_Q.$$

The result is an analog of the Ward identity $\partial_P \Sigma_P = \Gamma_{P,P,0}$ for systems with cubic interaction $\hat{c}^\dagger \hat{c}(\hat{a}^\dagger + \hat{a})$.





Consider positive sets $\{p_i\}, \{p'_i\}$. Then

$$\frac{\sum_i (\ln p'_i - \ln p_i) p'_i}{\sum_i p'_i} + \ln \sum_i p_i - \ln \sum_i p'_i \geq 0,$$

and the upperbound is reached at $p'_i = C p_i$.
Applied to stochastic ensemble, this inequality reads

$$\langle E' - E \rangle' + \ln \sum_i e^{-E'_i} - \ln \sum_i e^{-E_i} \geq 0.$$

The trial ensemble $\{E'_i\}$ should be varied to maximize $\langle E' - E \rangle' + \ln \sum_i e^{-E'_i}$.

Let us assume Gaussian trial statistics

$$S' = \bar{a}_i G^{-1}_{ij} a_j$$

and optimize G according to the Feynman's principle:

$$\frac{\delta}{\delta G}\left(\langle S' - S\rangle' + \ln \sum_i e^{-S'_i}\right) = 0.$$

It gives:

$$\delta \langle S\rangle' G_{\bar{1}1} = \delta G_{\bar{1}1} \qquad \text{or equally} \qquad \delta \langle S\rangle' G^{-1}_{1\bar{1}} = -\delta G^{-1}{}_{1\bar{1}}.$$

This equation is equivalent to the second Schwinger-Dyson equation

$$\left\langle a_{\bar{1}} \frac{\partial S}{\partial a_1} \right\rangle' = \delta_{\bar{1}1},$$

where the Gaussian statistics is assumed.

This chapter is inspired by a recent work with V.Vyborova

The second Schwinger-Dyson equation reads

$$\frac{\int a_1 e^{-\bar{a}G^{-1}a} \partial_{a_2} S[\bar{a}, a] D\bar{a}, a}{\int e^{-\bar{a}Ga} D\bar{a}, a} = \delta_{12}.$$

Integration over a_1 by parts gives

$$\left\langle \left(a_1 \bar{a}_{\bar{1}} G^{-1}_{\bar{1}2} - \delta_{12} \right) S \right\rangle' = \delta_{12},$$

or

$$\langle a_1 \bar{a}_{\bar{1}} S \rangle' - G_{1\bar{1}} \langle S \rangle' = G_{1\bar{1}}.$$

L.h.s. equals $-\frac{\delta \langle S \rangle'}{\delta G^{-1}_{\bar{1}1}}$. This completes the proof.