

Ultracold quantum Gases

Part 2 Degenerate Fermi gases

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Abstract

This is the second part of our lecturing course on ultracold quantum gases. It discusses ultracold degenerate gases of fermionic atoms and is focused on the role of interactions and on superfluidity. After creating quantum degenerate atomic Fermi gases, experiments have reached the so-called strongly interacting regime bringing analogies with neutron stars and high temperature superconductivity. Therefore, aside from an Introduction to the theory of degenerate fermions, the lecturing course includes several modern developments in this domain.

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Lecture 1. Key quantities. Elastic and inelastic interaction between atoms

1.1 Length and energy scales

Characteristic length scales for a gas of atomic fermions are the same as in the case of bosonic atoms. They have been introduced in *Part 1* of the lecturing course in section 1.1. So, there is the thermal de Broglie wavelength $\Lambda_T = (2\pi\hbar^2/mT)^{1/2}$, where m is the atom mass, and T the gas temperature, and this length scale is inversely proportional to the thermal wavevector of atoms, k_T . Then, there is a characteristic radius of interaction between atoms, R_e . The wavefunction of the atoms is influenced by the interaction only at interatomic distances of the order of or smaller than R_e , and at much larger distances their motion is free. The third length scale is the mean interatomic separation, $n^{-1/3}$, with n being the gas density.

The so-called dilute and ultracold limits have been also introduced in section 1.1 of *Part 1*. The dilute limit assumes that the mean interparticle distance is much larger than the characteristic radius of interaction, i.e. $n^{-1/3} \gg R_e$ or $nR_e^3 \ll 1$. Note that this condition is necessary for considering the system as a gas.

The ultracold limit relies on the condition $\Lambda_T \gg R_e$, i.e. it assumes that the de Broglie wavelength of particles greatly exceeds the radius of interaction

between them. In a dilute ultracold gas we usually consider only pair interactions and collisions between the particles. The inequality $\Lambda_T \gg R_e$ provides the dominant role of s -wave collisions, that is the collisions with zero orbital angular momentum. However, for identical fermions the s -wave scattering is not possible because the wavefunction of the relative motion should change sign under permutation of the fermions. Thus, only the scattering with odd orbital angular momenta is allowed, and the leading channel is the p -wave scattering, i.e. the scattering with orbital angular momentum $l = 1$. We will see that this leads to a crucial difference of fermions from bosons with respect to collisional properties.

As well as in the case of bosons, we introduce the degeneracy parameter $n\Lambda_T^3$. When the de Broglie wavelength Λ_T is much smaller than the mean interparticle separation $n^{-1/3}$ and hence $n\Lambda_T^3 \ll 1$, then the gas is classical and it obeys the Boltzmann statistics. In the case of $n\Lambda_T^3 \gtrsim 1$, which in principle can be achieved either by decreasing the temperature or by increasing the density, the gas becomes *degenerate* and *quantum statistics* comes into play. For fermions this leads to the formation of *Fermi sea*.

Let us consider a gas of non-interacting single-component fermions in free space (the term *single-component* means that all particles are in the same internal quantum state). Then the particle wavevector \mathbf{k} is a good quantum number and the mean occupation number of the state with a given \mathbf{k} is governed by the Fermi distribution:

$$N_k = \frac{1}{\exp\{(E_k - \mu)/T\} + 1}, \quad (1.1)$$

where μ is the chemical potential, and $E_k = \hbar^2 k^2/2m$ is the single particle kinetic energy. The dependence of the chemical potential on T and n follows from the normalization condition:

$$N = \sum_{\mathbf{k}} N_k = \int N_k \frac{V d^3 k}{(2\pi)^3}, \quad (1.2)$$

with N being the total number of particles, and V the volume of the system.

For $T = 0$ one has to have $\mu_0 \equiv \mu(T = 0) > 0$ and

$$\begin{aligned} N_k &= 0; & \text{for } E_k > \mu_0, \\ N_k &= 1; & \text{for } E_k < \mu_0. \end{aligned} \quad (1.3)$$

In other words, N identical fermions in the ground state occupy N lowest quantum states, so that there is one fermion in each such state. This is clear from the idea of Fermi statistics, where due to the Pauli principle there is not more than one fermion in a given quantum state. Then, equation (1.2) at $T = 0$ takes the form:

$$N = \int_0^{k_F} V \frac{4\pi k^2 dk}{8\pi^3} = \frac{V k_F^3}{6\pi^2}, \quad (1.4)$$

where the quantities $\hbar^2 k_F^2/2m = \mu_0 \equiv E_F$ and $\hbar k_F$ are called *Fermi energy* and *Fermi momentum*.

The zero-temperature distribution function N_k has a stepwise behavior and is shown in black in Fig 1.1. At a finite T such that $T \ll E_F$, the mean occupation number N_k is smaller than 1 at $k < k_F$, and there are occupied states ($N_k > 0$) at $k > k_F$. The corresponding distribution function is given by the blue curve in Fig. 1.1.

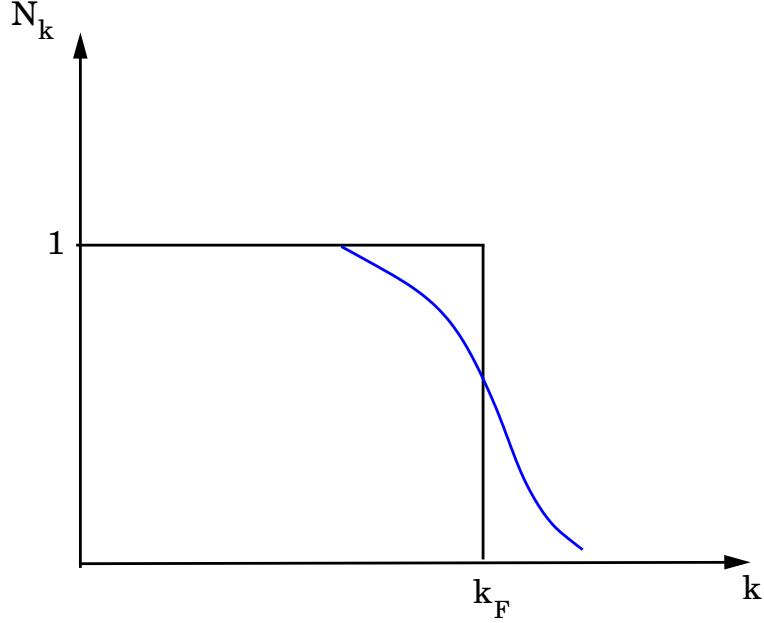


Figure 1.1: The distribution function N_k as a function of the fermion momentum k at zero temperature (in black) and at a finite temperature $T \ll E_F$ (in blue).

From Eq. (1.4) we obtain the Fermi momentum as a function of the density at $T = 0$:

$$\hbar k_F = \hbar \left(\frac{6\pi^2 N}{V} \right)^{1/3} = \hbar (6\pi^2 n)^{1/3}. \quad (1.5)$$

Accordingly, for the Fermi energy we have:

$$E_F = \frac{(6\pi^2)^{2/3} \hbar^2 n^{2/3}}{2m}. \quad (1.6)$$

The de Broglie wavelength of particles with energies $\sim E_F$ is

$$\Lambda_F = \left(\frac{2\pi\hbar^2}{mE_F} \right)^{1/2} = n^{-1/3} \left(\frac{4}{3\sqrt{\pi}} \right)^{1/3}. \quad (1.7)$$

So, we have $n\Lambda_F^3 \sim 1$, which is obvious as the Fermi energy is of the order of the temperature of quantum degeneracy. At $T \lesssim E_F$ the Fermi gas is degenerate

and for this reason the distribution function displayed in Fig. 1.1 is drastically different from that for the classical gas. It is called the *Fermi sea*.

We thus see that without interactions there are two energy scales in the Fermi gas, temperature T and Fermi energy E_F . The first quantum degenerate Fermi gas has been obtained at JILA in 1998 (D. Jin, ^{40}K), and presently tens of labs all over the world are working with degenerate fermions. As well as ultracold bosonic atoms, they are created by evaporative and optical cooling in magnetic and optical traps as discussed in the course of J.T.M. Walraven. Ongoing experiments are commonly dealing with temperatures from 100 nK to 1 μK and densities ranging from 10^{12} to 10^{14} cm^{-3} . The number of fermionic atoms is usually in the range from 10^4 to 10^7 .

1.2 Interaction between identical fermions

The third energy scale is established by the interaction between particles. In the dilute limit where the condition $nR_e^3 \ll 1$ is satisfied, the total interaction energy in the system is the sum of all pair interactions, $E_{int} = (N^2/2)\epsilon_{int}$, where $N \gg 1$ is the number of particles, $N^2/2$ is the number of pairs, and ϵ_{int} is the interaction energy for a pair of atoms. Two important circumstances should be mentioned at this point. First of all, we consider here elastic interaction between particles, i.e. the interaction which does not change their internal states. Second, we assume the regime of weak interactions, where the interaction between particles can be taken into account in a sort of many-body perturbative approach. The related criterion will be discussed later in this *Lecture*.

The quantity ϵ_{int} has been calculated in section 1.2 of *Part 1* of the course. It is expressed through the phase shift (amplitude) of scattering of two particles with relative wavevector k and orbital angular momentum l . The derivation in section 1.2 of *Part 1* is given for the s -wave scattering ($l = 0$), but it remains exactly the same for $l \neq 0$. So, for given values of k and l we have:

$$\epsilon_{int}(l, k) = \frac{\langle g_l(k) \rangle}{V}; \quad g_l(k) = -\frac{4\pi\hbar^2}{m} \left(\frac{\delta_l(k)}{k} \right), \quad (1.8)$$

where $\delta_l(k)$ is the scattering phase shift for the orbital angular momentum l , and the symbol $\langle \rangle$ stands for the average over the momentum distribution of particles. Note that equation (1.8) does not take into account correlations between particles and for momentum-dependent g_l a special care should be taken in this respect.

The scattering phase shift follows from the condition that in the limit of interatomic separations $r \rightarrow \infty$ the wavefunction of the relative motion of two atoms with orbital angular momentum l takes the form

$$\psi_l(r) \sim \frac{\sin(kr - \pi l/2 + \delta_l)}{kr}.$$

The calculation of δ_l relies on the knowledge of the potential of interaction between the atoms, $U(r)$. This is done in *Problem 1.1* to this lecture. The

interaction potential $U(r)$ for the atoms has a Van der Waals long-range tail, i.e. it decays at large distances as $1/r^6$. In this case, for identical fermions the leading interaction (scattering) channel is the p -wave scattering and away from resonances the corresponding phase shift is

$$\delta_1(k) = -\frac{(kb)^3}{3}, \quad (1.9)$$

where b is a characteristic distance which depends on the shape of $U(r)$, and we assume the inequality $k|b| \ll 1$.

For small k the p -wave scattering phase shift is much smaller than the phase shift for the s -wave scattering. The latter is given by $\delta_0 = -ka$, where a is the so-called scattering length. This is because for $l = 1$ the centrifugal barrier reduces the probability for particles to approach each other at short distances where the potential $U(r)$ provides an efficient scattering (see *Problem 1.1* to this lecture).

In the case of identical fermions, using equations (1.8) and (1.9) we obtain

$$g(k) = \frac{4\pi\hbar^2}{3m}(k^2b^3). \quad (1.10)$$

The total interaction energy then becomes

$$E_{int} = \frac{N^2}{2V}\langle g(k) \rangle = \frac{4\pi\hbar^2b^3}{3mV}\langle k^2 \rangle \frac{N^2}{2}, \quad (1.11)$$

and the interaction energy per particle is given by

$$\frac{\partial E_{int}}{\partial N} = n\langle g(k) \rangle, \quad (1.12)$$

assuming that $\langle g(k) \rangle$ is independent of the density. Note, however, that the presence of this dependence only changes the numerical coefficient.

At $T \ll E_F$ characteristic momenta of fermions are of the order of $\hbar k_F$ and from Eqs. (1.10) and (1.12) we find the interaction energy per particle $n\langle g(k) \rangle \sim E_F(nb^3)$. In the opposite limit of a classical Fermi gas, $T \gg E_F$, where characteristic particle momenta are of the order of the thermal momentum $\hbar k_T = \hbar(mT/\hbar^2)^{1/2}$, we obtain $n\langle g(k) \rangle \sim T(nb^3)$. The atomic distance $|b|$ is generally of the order of R_e , and hence in the dilute limit we have the inequality $n|b|^3 \ll 1$. We thus see that at both low and high temperatures the interaction energy is smaller than the kinetic energy of particles by a factor of $n|b|^3$. At realistic densities $n \sim 10^{12} - 10^{14} \text{ cm}^{-3}$ this factor is ranging from 10^{-9} to 10^{-5} (away from p -wave resonances) and therefore the interaction between identical fermions is usually omitted.

1.3 Elastic and inelastic collisions

The characteristic distance b is directly related to all parameters of elastic p -wave scattering. The two-body scattering problem has been discussed in detail

in section 1.3 of *Part 1* of the course. This problem is equivalent to the scattering of a particle with mass $m/2$ from a force center, due to the particle-center interaction via the potential $U(r)$. As a result of the interaction, an incident particle moving along the axis z with momentum \mathbf{k} is scattered on an angle θ and acquires the momentum \mathbf{k}' (see section 1.3 of *Part 1*). For elastic scattering we have $|\mathbf{k}'| = |\mathbf{k}|$. At an infinite separation from the center the particle wavefunction is a superposition of the incident plane wave and scattered spherical wave:

$$\psi = \exp(ikz) + \frac{f(\theta)}{r} \exp(ikr), \quad (1.13)$$

where $f(\theta)$ is called the scattering amplitude. The probability $\alpha(k)$ for the scattered wave to pass through the surface of a sphere of radius r per unit time is equal to the intensity of the scattered wave $|f(\theta)|^2/r^2$ multiplied by $vr^2d\Omega$ and integrated over the solid angle Ω . Here $v = 2\hbar k/m$ is the velocity of the incident particle or, returning to the two-body scattering problem, the relative velocity of colliding atoms. We thus have $\alpha(k) = \int v|f(\theta)|^2d\Omega$, and this quantity is called the rate constant of elastic collisions. The number of scattering events per unit time and unit volume is given by $\bar{\alpha}N^2/2V$, where $\bar{\alpha}$ is the rate constant averaged over the momentum distribution of particles. The number of collisions that a given particle experiences per unit time is $\bar{\alpha}n$, and the quantity $(\bar{\alpha}n)^{-1}$ can be identified as a characteristic collisional or kinetic time τ_K . The quantity $\sigma(k) = \alpha(k)/v = \int |f(\theta)|^2d\Omega$ has a dimension of surface area and is called the elastic cross section. The mean free path of a particle is $\lambda \sim 1/(n\sigma)$.

For finding the scattering amplitude $f(\theta)$ one commonly expands it in Legendre polynomials:

$$f(\theta) = \sum_{l=0}^{\infty} (2l+1) f_l P_l(\cos\theta),$$

where f_l are called partial scattering amplitudes. They correspond to the scattering with orbital angular momentum l . The solution of the two-body scattering problem in section 1.3 of *Part 1* and in *Problem 1.1* to this lecture establish a relation between the partial scattering amplitude f_l and the scattering phase shift δ_l :

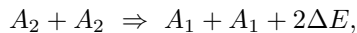
$$f_l = \frac{\tan \delta_l}{k(1 - i \tan \delta_l)}.$$

For $k|b| \ll 1$, using Eq. (1.9) we immediately obtain the p -wave scattering amplitude

$$f_1 = -\frac{1}{3}b(kb)^2. \quad (1.14)$$

This amplitude is actually very small. Since the parameter b is generally of the order of R_e , we have $|f_1| \ll R_e$. The scattering cross section is $\sigma_1 \sim b^2(kb)^4$ and in the ultracold limit it is at least several orders of magnitude smaller than R_e^2 . The rate of elastic collisions is $\alpha_1 = \sigma_1 v \propto k^5$ and is generally very slow. For realistic parameters in cold gases we have the collisional time $\tau_K = (\bar{\alpha}n)^{-1}$ exceeding seconds. Therefore, elastic collisions between identical fermions usually do not play an important role.

We now discuss inelastic collisions, i.e. collisions changing the internal states of the colliding atoms. Consider a gas of atoms which can be in two different internal states, the ground state A_1 and an excited (hyperfine) state A_2 . Then, binary collisions involving atoms in the state A_2 can lead to their relaxation transition to the ground state. One may consider, for example, the following inelastic collisional process:



where ΔE is the difference between energies of the internal states A_2 and A_1 . Usually in cold gases ΔE greatly exceeds the kinetic energy of colliding atoms, so that the final state of the pair is independent of the initial collision energy (momentum). In this respect, inelastic collisions in cold gases can be identified as "deep inelastic processes".

Inelastic collisions involving the s -wave in the incoming channel have been discussed in section 1.4 of *Part 1* of the course. For collisions of identical fermions only odd angular momenta l in the incoming channel are possible and usually the p -wave ($l = 1$) is the most important. This makes inelastic collisions rather slow in the dilute ultracold limit, which is already seen without any calculations. Consider two fermions at large separations \mathbf{r} from each other, where the interaction potential $U(r)$ is no longer important. Then the wavefunction of their relative motion is simply an antisymmetrized superposition of plane waves

$$\psi = \frac{1}{\sqrt{2}} [\exp(i\mathbf{k}\mathbf{r}) - \exp(-i\mathbf{k}\mathbf{r})].$$

In the ultracold limit the inequality $kR_e \ll 1$ is satisfied, and there is a large interval of distances where $R_e \ll r \ll 1/k$ and the interaction is still not important. In this interval, expanding the exponents in powers of $(\mathbf{k}\mathbf{r})$ we obtain $\psi \propto (\mathbf{k}\mathbf{r})$. So, we may say that "cold identical fermions do not like to approach each other". The dependence $\psi \propto k$ is preserved with decreasing the interatomic separation r and making it of the order of or smaller than R_e . The interaction only changes the coordinate dependence of ψ irrespective of the value of k . The rate of an inelastic decay process is proportional to the square of the transition matrix element of an inelastic interaction between the initial and final states. For deep inelastic processes, the wavefunction and energy of the final state are practically independent of the initial momentum k . Thus, the only quantity that depends on k is the initial wavefunction ψ , which immediately shows that the inelastic rate is $\alpha_{in} \propto k^2$.

This conclusion is confirmed by calculations in *Problem 1.2* to this lecture. In general, the rate of inelastic processes depends on the shape of the interaction potential $U(r)$ and on the interaction providing inelastic transitions. In order to obtain the dependence of the inelastic rate on the relative momentum k of colliding atoms we simplified the problem and assumed that the internal states of the atoms are completely changed once the interatomic distance reaches a certain value r_0 . At the same time it is assumed that inelastic transitions do not occur at $r > r_0$. This is equivalent to putting a perfectly absorbing wall

at $r = r_0$, so that there is only an incoming spherical flux at this point (see *Problem 1* to this *Lecture*).

As the rate of inelastic collisions between identical fermions is proportional to k^2 , it is usually not very important in ultracold gases.

1.4 Two-component Fermi gas. Weakly interacting regime.

The situation with interparticle interaction drastically changes in a two-component Fermi gas, i.e. when fermionic atoms in two different internal states are present. The interaction between fermions of different components originates from their s -wave scattering. The phase shift for the s -wave scattering is given by $\delta_0 = -ka$ assuming that $k|a| \ll 1$, where the quantity a is called the scattering length and it depends on the shape of the potential $U(r)$ (see section 1.2 of *Part 1*). The s -wave scattering amplitude is $f_0 = -a$ as follows from equation (1.37) in *Part 1* at $k|a| \ll 1$ and *Problem 1.1* to this lecture. Accordingly, the contribution of the intercomponent interaction to the interaction energy in the system is

$$E_{int}^0 = \frac{4\pi\hbar^2 a}{mV} N_1 N_2, \quad (1.15)$$

where the first multiple is the interaction energy for an atomic pair consisting of fermions in different internal states, and the second multiple represents the number of such pairs, with N_1 and N_2 being the number of particles in the first and second component.

Let us now assume that $N_1 = N_2 = N$ and compare the energy of the intercomponent interaction (1.15) with the energy of interaction between fermions in the same internal state (intra-component interaction). The latter originates from the p -wave scattering and follows from Eq. (1.11):

$$E_{int}^1 = \frac{4\pi\hbar^2 b^3}{3mV} \langle k^2 \rangle N^2, \quad (1.16)$$

where N^2 appears as $(N_1^2/2 + N_2^2/2)$ for $N_1 = N_2 = N$. On the other hand, for $N_1 = N_2 = N$ equation (1.15) gives

$$E_{int}^0 = \frac{4\pi\hbar^2 a}{mV} N^2. \quad (1.17)$$

Generally, we have $|a| \sim |b| \sim R_e$ and in the ultracold dilute limit the ratio of the intra- to intercomponent interaction energy is

$$\frac{E_{int}^1}{E_{int}^0} \sim \langle k^2 \rangle R_e^2 \ll 1.$$

The intercomponent interaction leads to the interaction energy per particle

$$ng = \frac{4\pi\hbar^2 a}{m} n \sim E_F(k_F a) \sim E_F(na^3)^{1/3} \quad (1.18)$$

and it can no longer be omitted. Moreover, as we will see later in the course, this interaction leads to all interesting physics in dilute Fermi gases.

Usually, the interaction between particles in quantum gases is treated within the many-body perturbation theory. For Fermi gases this assumes that the interaction energy per particle is much smaller than the characteristic kinetic energy of particles. In the case of quantum degenerate fermions, the latter is $\sim E_F$. Then, from Eq. (1.18) we immediately find the condition of such *weakly interacting* regime:

$$k_F|a| \ll 1 \rightarrow (n|a|^3)^{1/3} \ll 1. \quad (1.19)$$

This condition should be certainly completed by the criterion of the dilute limit, $nR_e^3 \ll 1$, which is especially important if for a given potential $U(r)$ the scattering length a turns out to be anomalously small.

Problems 1

1.1 Calculate the phase shift and scattering amplitude for elastic scattering with orbital angular momentum l in the limit of low collision energies. Show that the scattering with $l > 0$ is much weaker than the s -wave scattering ($l = 0$).

The scattering problem for two particles with equal masses m is equivalent to the scattering of a particle with mass $m/2$ from the force center, due to the particle-center interaction via the potential $U(r)$. The Schroedinger equation for the relative motion of these particles then reads:

$$\left[-\frac{\hbar^2}{m}\Delta_{\mathbf{r}} + U(r) \right] \psi(\mathbf{r}) = E\psi(\mathbf{r}), \quad (1.20)$$

where $E = \hbar^2k^2/m$ is the collision energy. We now expand the wavefunction of the relative motion of colliding atoms, $\psi(\mathbf{r})$ in Legendre polynomials:

$$\psi(\mathbf{r}) = \sum_l (2l+1)P_l(\cos\theta)\psi_l(r), \quad (1.21)$$

where θ is the scattering angle, and $\psi_l(r)$ is the wavefunction of the relative motion with orbital angular momentum l . Since we took the z axis (the axis along the wave vector of the incident plane wave) as the axis of quantization, due to the axial symmetry of our scattering problem the wavefunction $\psi(\mathbf{r})$ (1.21) is independent of the azimuthal angle ϕ . Substituting $\psi(\mathbf{r})$ (1.21) into the Schroedinger equation (1.20), multiplying both sides of the equation by $P_l(\cos\theta)$ and integrating over the solid angle we obtain an equation for the relative motion with a given orbital angular momentum l :

$$-\frac{\hbar^2}{m} \left[\frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} - \frac{l(l+1)}{r^2} \right] \psi_l(r) + U(r)\psi_l(r) = \frac{\hbar^2k^2}{m}\psi_l(r). \quad (1.22)$$

When obtaining Eq. (1.22) we took into account the well-known expression for the Laplacian in spherical coordinates:

$$\Delta_{\mathbf{r}} = \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \left(\frac{\partial^2}{\partial \theta^2} + \cot \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right)$$

and the fact that

$$\left(\frac{d^2}{d\theta^2} + \cot \theta \frac{d}{d\theta} \right) P_l(\cos \theta) = -l(l+1)P_l(\cos \theta).$$

Equation (1.22) can be rewritten in the form:

$$-\frac{\hbar^2}{m} \left[\frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} \right] \psi_l(r) + U_{eff}(r)\psi_l(r) = \frac{\hbar^2 k^2}{m} \psi_l(r), \quad (1.23)$$

with

$$U_{eff}(r) = U(r) + \frac{\hbar^2 l(l+1)}{mr^2} \quad (1.24)$$

being the so-called effective potential (see Fig. 1.2).

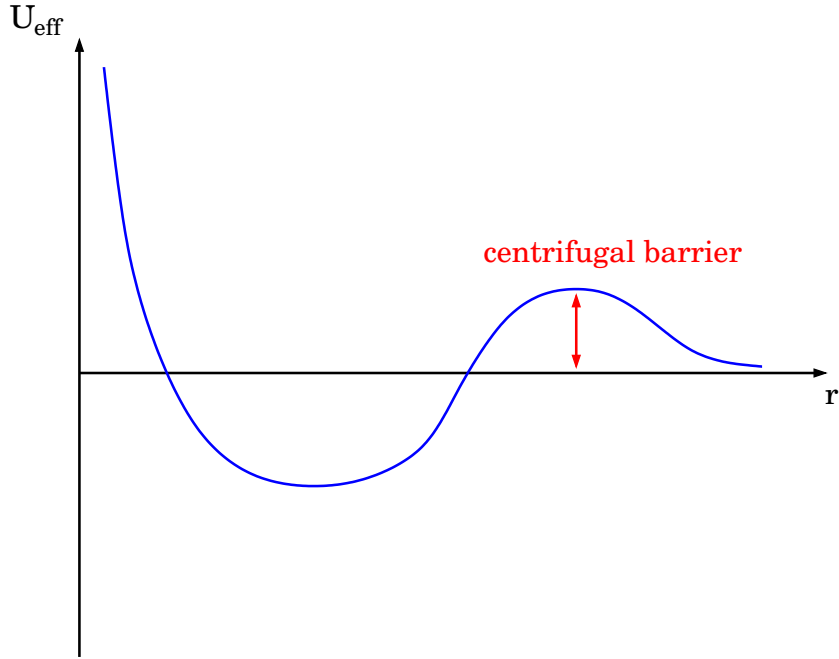


Figure 1.2: Effective potential $U_{eff}(r) = U(r) + \hbar^2 l(l+1)/mr^2$ for the relative motion with orbital angular momentum $l > 0$.

At distances $r \gg R_e$ we can drop $U(r)$ from Eq. (1.22), which transforms it to the equation of free motion with orbital angular momentum l . The solution then reads:

$$\psi_l \propto [j_l(kr) + C\eta_l(kr)], \quad (1.25)$$

where the constant C is independent of k . The spherical Bessel and Neumann functions, j_l and η_l , have the following asymptotic behavior at large arguments:

$$j_l(x) \Rightarrow \frac{\sin(x - \pi l/2)}{x}; \quad x \rightarrow \infty \quad (1.26)$$

$$\eta_l(x) \Rightarrow -\frac{\cos(x - \pi l/2)}{x}; \quad x \rightarrow \infty \quad (1.27)$$

The scattering phase shift δ_l is defined by writing ψ_l at $r \rightarrow \infty$ as

$$\psi_l \propto \frac{\sin(kr - \pi l/2 + \delta_l)}{kr}, \quad r \rightarrow \infty. \quad (1.28)$$

For $U(r) = 0$ we have $\delta_l = 0$ and $\psi_l \propto j_l(kr)$. Comparing Eq. (1.28) with Eq. (1.25) we obtain:

$$\tan \delta_l = -C. \quad (1.29)$$

At distances where $R_e \ll r \ll k^{-1}$, we may drop both $U(r)$ and $\hbar^2 k^2/m$ from the Schroedinger equation (1.22) and then obtain

$$\psi_l \propto \left(r^l - \frac{b^{2l+1}}{r^{l+1}} \right), \quad (1.30)$$

where the length b is independent of k . Importantly, we assume that $U(r)$ decays faster than r^{-2l-3} and, hence, the term b^{2l+1}/r^{2l+3} originating from the kinetic energy exceeds the potential energy term $U(r)r^l$. Within a normalization coefficient, Eq. (1.30) should coincide with ψ_l following from Eq. (1.25) at $kr \ll 1$. Taking into account asymptotic expressions for the spherical Bessel and Neumann functions at small arguments:

$$j_l(x) \Rightarrow \frac{\sqrt{\pi}}{2^{l+1}\Gamma(l+3/2)} x^l; \quad x \ll 1, \quad (1.31)$$

$$\eta_l(x) \Rightarrow (-1)^l \frac{2^l \sqrt{\pi}}{\Gamma(1/2-l)} x^{-l-1}; \quad x \ll 1, \quad (1.32)$$

we reduce ψ_l (1.25) at $kr \ll 1$ to

$$\psi_l \propto \left[\frac{\sqrt{\pi}}{2^{l+1}\Gamma(l+3/2)} (kr)^l + (-1)^l C \frac{2^l \sqrt{\pi}}{\Gamma(1/2-l)} (kr)^{-l-1} \right].$$

The comparison of this expression with Eq. (1.30) immediately gives

$$C = (-1)^l \left(\frac{bk}{2} \right)^{2l+1} \frac{\Gamma(1/2-l)}{\Gamma(l+3/2)}, \quad (1.33)$$

and Eq. (1.29) yields

$$\tan \delta_l = (-1)^{l+1} \left(\frac{kb}{2} \right)^{2l+1} \frac{\Gamma(1/2-l)}{\Gamma(l+3/2)}. \quad (1.34)$$

Using the expansion of the scattering amplitude in Legendre polynomials given by Eq. (1.21) and recalling that at $r \rightarrow \infty$ the total wavefunction ψ is given by Eq. (1.13), we write the l -wave part of ψ as

$$\psi_l = i^l \frac{\sin(kr - \pi l/2)}{kr} + \frac{f_l}{r} \exp(ikr)$$

and after a certain algebra transform it to

$$\psi_l \propto \frac{1}{kr} \{(1 + ikf_l) \sin(kr - \pi l/2) + kf_l \cos(kr - \pi l/2)\}.$$

Since this expression should coincide within a normalization coefficient with ψ_l (1.28) we immediately find a relation between the scattering phase shift and scattering amplitude:

$$f_l = \frac{\tan \delta_l}{k(1 - i \tan \delta_l)} \equiv \frac{\exp(2i\delta_l) - 1}{2ik}. \quad (1.35)$$

In the limit of low collision energies we put $k|b| \ll 1$ and obtain;

$$f_l = \frac{\delta_l}{k} = (-1)^{l+1} \frac{b}{2} \left(\frac{kb}{2}\right)^{2l} \frac{\Gamma(1/2 - l)}{\Gamma(l + 3/2)}. \quad (1.36)$$

Then the l -wave scattering cross section is

$$\sigma_l = 4\pi(2l + 1)|f_l|^2 = \frac{4\pi}{k^2}(2l + 1) \sin^2 \delta_l \propto k^{4l}. \quad (1.37)$$

We thus see that for slow particles the l -wave scattering (away from l -wave resonances) is much weaker than the s -wave scattering. The physical reason is that the centrifugal barrier of the effective potential $U_{eff}(r)$ reduces the amplitude of the wavefunction at short distances where the interaction potential $U(r)$ acts.

1.2 Calculate the inelastic rate constant for collisions of ultracold atoms interacting with each other via the potential $U(r)$ representing a perfectly absorbing wall at $r = r_0$ and having a deep potential well for $r_0 < r < R_0$ (see Fig. 1.3).

We start with Eq. (1.23) for the wavefunction $\psi_l(r)$ of the relative motion of atoms with orbital angular momentum l . We assume that the collision energy E_k and the centrifugal potential $\hbar^2 l(l+1)/mr_0^2$ are much smaller than the well depth U_0 . Then, for $r_0 < r < R_0$ the solution of Eq. (1.23) reads:

$$\psi_l(r) = A\{j_l(k_0 r) + B h_l(k_0 r)\}, \quad (1.38)$$

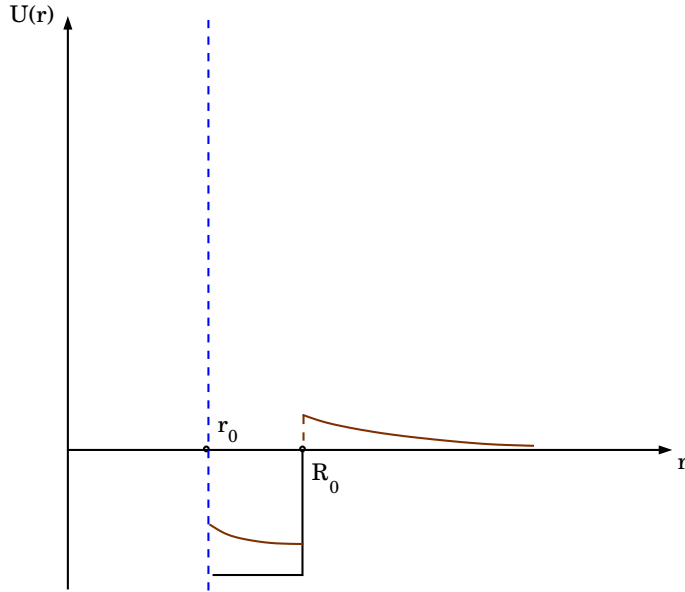


Figure 1.3: The interaction potential $U(r)$ having a perfectly absorbing wall at $r = r_0$ and a deep potential well for $r_0 < r < R_0$. The brown curve shows the effective potential $U_{eff}(r) = U(r) + \hbar^2 l(l+1)/mr^2$.

where j_l and h_l are spherical Bessel and Hankel functions, $k_0 = \sqrt{mU_0}/\hbar$, and the coefficients A and B have to be determined. The presence of the perfectly absorbing wall at $r = r_0$ means that for $r \rightarrow r_0$ we only have an incoming spherical flux and hence $\psi_l(r) \propto \exp[-ik_0(r - r_0)]$. The assumption $\hbar^2 l(l+1)/mr_0^2 \ll U_0$ is equivalent to $k_0 r_0 \gg 1$, i.e. the arguments of the Bessel and Hankel functions in Eq. (1.38) are large. This allows us to use the asymptotic expression (1.26) and a similar asymptotic expression for h_l :

$$h_l(x) \Rightarrow -i \frac{\exp(ix - i\pi l/2)}{x}; \quad x \rightarrow \infty. \quad (1.39)$$

Then, in order to satisfy the boundary condition for $r \rightarrow r_0$ we should put $B = -1/2$, and the wavefunction in the interval of distances $r_0 < r < R_0$ takes the form

$$\psi_l(r) = \frac{iA}{2k_0 r} \exp(-ik_0 r + i\pi l/2). \quad (1.40)$$

For $r > R_0$ the motion is free and we write the solution as

$$\psi_l(r) = i^l [j_l(kr) + ik f_l h_l(kr)] \quad (1.41)$$

so that at $r \rightarrow \infty$ the second term in Eq. (1.41) becomes $(f_l/r) \exp(ikr)$ and thus describes an outgoing spherical wave with orbital angular momentum l ,

with f_l being the l -wave scattering amplitude. Note that multiplying Eq. (1.41) by $(2l+1)P_l(\cos\theta)$, making the summation over l , and using the relations

$$\exp(ikz) = \sum_l i^l (2l+1) j_l(kr) P_l(\cos\theta); \quad f(\theta) = \sum_l (2l+1) f_l P_l(\cos\theta),$$

where $f(\theta)$ is the total scattering amplitude, we obtain the required expression for the wavefunction $\psi(\mathbf{r})$ at $r \rightarrow \infty$:

$$\psi(\mathbf{r}) = \exp(ikz) + \frac{f(\theta)}{r} \exp(ikr).$$

In the ultracold limit we have $kR_0 \ll 1$ and at r close to R_0 in Eq. (1.41) we may use the asymptotic expression (1.31) and the asymptotic expression

$$h_l(x) \Rightarrow \frac{i(-1)^{l+1} \sqrt{\pi}}{\Gamma(1/2-l)} \frac{2^l}{x^{l+1}}; \quad x \ll 1. \quad (1.42)$$

Then, equalizing the wavefunctions (1.41) and (1.40) and their derivatives at $r = R_0$ and using the inequality $k_0 R_0 \gg 1$, we obtain:

$$f_l = \frac{i j_l(kR_0)}{k h_l(kR_0)} \left[1 - \frac{(2l+1)i}{k_0 R_0} \right] = \frac{(-1)^{l+1} \Gamma(1/2-l) (kR_0)^{2l}}{\Gamma(l+3/2) 2^{2l+1}} \left[1 - \frac{(2l+1)i}{k_0 R_0} \right] R_0, \quad (1.43)$$

$$|A| = 2(2l+1) j_l(kR_0) = \frac{\sqrt{\pi} (2l+1)}{2^l \Gamma(l+3/2)} (kR_0)^l. \quad (1.44)$$

The inelastic rate constant is equal to the incoming flux at $r \rightarrow r_0$, multiplied by the surface area $4\pi r_0^2$:

$$\alpha_{in} = \frac{\hbar}{im} \left\{ \left(\psi_l \frac{d\psi_l^*}{dr} - \psi_l^* \frac{d\psi_l}{dr} \right) \times 4\pi r^2 \right\}_{r \rightarrow r_0}. \quad (1.45)$$

Using ψ_l (1.40) with $|A|$ (1.44), we obtain

$$\alpha_{in} = \frac{2\pi\hbar|A|^2}{mk_0} = \frac{2\pi^2\hbar}{mk_0} \left[\frac{(2l+1)}{\Gamma(l+3/2)} \right]^2 \left(\frac{kR_0}{2} \right)^{2l}. \quad (1.46)$$

For identical fermions only odd l are possible, and the leading decay channel is related to the p -wave scattering ($l=1$). Thus, we have

$$\alpha_{in} \simeq \frac{8\pi\hbar}{mk_0} (kR_0)^2 \propto k^2. \quad (1.47)$$

Taking into account that for identical fermions the wavefunction should be antisymmetrized, we have to multiply Eq. (1.47) by a factor of 2.

The obtained result (1.47) justifies the dimensional estimate $\alpha_{in} \propto k^2$ obtained in subsection 1.3.

1.3. Find the l -wave scattering amplitude and the inelastic rate constant in the limit of $k \rightarrow 0$ for the scattering potential which has the form $U(r) = -\beta/r^4$ at $r \geq r_0$ and has a perfectly absorbing wall at $r = r_0$ (see Fig. 1.4). Assume the validity of the WKB approximation at $r \rightarrow r_0$.

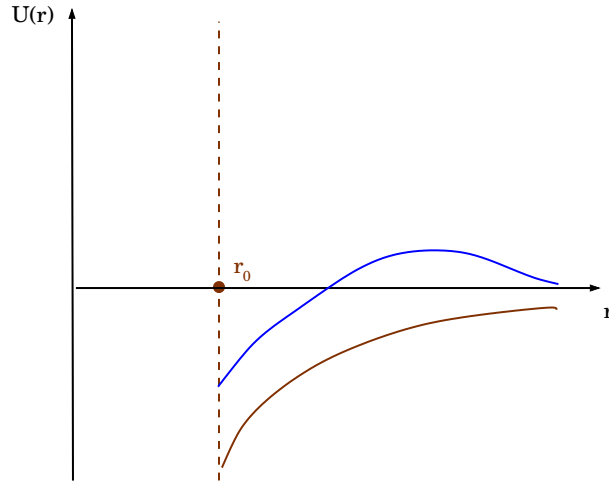


Figure 1.4: The scattering potential $U(r) = -\beta/r^4$ ($r > r_0$) with a perfectly absorbing wall at $r = r_0$ (brown curve). The effective potential $U_{eff}(r) = U(r) + \hbar^2 l(l+1)/mr^2$ is shown by the blue curve.

Lecture 2. Second quantization. Ideal Fermi gas

2.1 Second quantization

We now introduce the method of second quantization for fermions. For simplicity we first consider N identical fermions in a large but finite volume and assume that they do not interact with each other, so that the Hamiltonian is

$$\hat{H}_{kin} = -\frac{\hbar^2}{2m} \sum_{\alpha} \frac{\partial^2}{\partial \mathbf{r}_{\alpha}^2}, \quad (2.1)$$

where particles are labelled by the index α and \mathbf{r}_{α} are their coordinates. The many-body wavefunction is antisymmetric with respect to an interchange of fermionic particles and has the form of a Slater determinant:

$$\psi = \frac{1}{\sqrt{N!}} \begin{vmatrix} \psi_{p_1}(\mathbf{r}_1) & \psi_{p_1}(\mathbf{r}_2) & \dots & \psi_{p_1}(\mathbf{r}_N) \\ \psi_{p_2}(\mathbf{r}_1) & \psi_{p_2}(\mathbf{r}_2) & \dots & \psi_{p_2}(\mathbf{r}_N) \\ \vdots & \vdots & \dots & \vdots \\ \psi_{p_N}(\mathbf{r}_1) & \psi_{p_N}(\mathbf{r}_2) & \dots & \psi_{p_N}(\mathbf{r}_N) \end{vmatrix} \quad (2.2)$$

where $\psi_{p_i}(\mathbf{r}_k)$ are wavefunctions of single-particle states. In order to establish the sign of ψ we numerate the states by the numbers

$$p_1 < p_2 < p_3 < \dots < p_N. \quad (2.3)$$

All p_i (2.3) are different from each other so that the occupation numbers N_i can be only 1 or 0.

Consider now an operator

$$\hat{F}^{(1)} = \sum_{\alpha} \hat{f}(\mathbf{r}_{\alpha}). \quad (2.4)$$

Its matrix elements will be non-zero only between the states with the same N_1, N_2, \dots or between the states in which one of these numbers increases by 1 and another one decreases by 1. For the diagonal matrix element we have:

$$\bar{F}^{(1)} = \sum_i f_{ii}^{(1)} N_i, \quad (2.5)$$

as well as in the case of bosons, and the notation

$$f_{ik}^{(1)} = \int \psi_i^*(\mathbf{r}) f^{(1)}(\mathbf{r}) \psi_k(\mathbf{r}) d^3r \quad (2.6)$$

is again used for the matrix element between the single particle states i and k . For the matrix element between the many-body states $1_i, 0_k$ and $0_i, 1_k$ (in the initial state $N_k = 1, N_i = 0$ and in the final state $N_k = 0, N_i = 1$, and it is assumed that $i < k$) we have

$$\langle 1_i, 0_k | F^{(1)} | 0_i, 1_k \rangle = f_{ik}^{(1)} (-1)^{A_{ik}}, \quad (2.7)$$

where A_{ik} is the sum of the occupation numbers for the states from $i + 1$ to $k - 1$:

$$A_{ik} = \sum_{j=i+1}^{k-1} N_j. \quad (2.8)$$

For $i = k - 1$ one should put $A_{ik} = 0$.

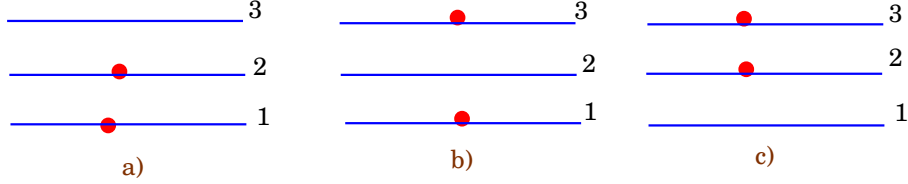


Figure 2.1: Two identical fermions (red balls) in three possible single-particle states (blue lines). The configurations from left to right show two-particle states a), b), and c) (see text).

As an example, consider 2 identical fermions and assume that each of them can be in one of the 3 states as depicted in Fig. 2.1. There are 3 two-particle states:

$$(a) \quad \frac{1}{\sqrt{2}} \{ \psi_1(\mathbf{r}_1) \psi_2(\mathbf{r}_2) - \psi_1(\mathbf{r}_2) \psi_2(\mathbf{r}_1) \}, \quad (2.9)$$

$$(b) \quad \frac{1}{\sqrt{2}} \{ \psi_1(\mathbf{r}_1) \psi_3(\mathbf{r}_2) - \psi_1(\mathbf{r}_2) \psi_3(\mathbf{r}_1) \}, \quad (2.10)$$

$$(c) \quad \frac{1}{\sqrt{2}} \{ \psi_2(\mathbf{r}_1) \psi_3(\mathbf{r}_2) - \psi_2(\mathbf{r}_2) \psi_3(\mathbf{r}_1) \}. \quad (2.11)$$

Matrix elements of the operator $\hat{F} = \hat{f}(\mathbf{r}_1) + \hat{f}(\mathbf{r}_2) + \hat{f}(\mathbf{r}_3)$ between these states are:

$$\begin{aligned} F_{ba} &= f_{32} \\ F_{ca} &= -f_{31} \\ F_{cb} &= f_{21} \\ F_{aa} &= f_{11} + f_{22} \\ F_{bb} &= f_{11} + f_{33} \\ F_{cc} &= f_{22} + f_{33}. \end{aligned}$$

One can easily check that these relations coincide with what we get from Eqs. (2.5) and (2.7).

We now turn to the operators \hat{a}_i which act not on the functions of particle coordinates, but on the functions of occupation numbers:

$$\hat{a}_i |N_1, N_2, \dots, N_i, \dots\rangle \Rightarrow |N_1, N_2, \dots, N_i - 1, \dots\rangle, \quad (2.12)$$

$$\hat{a}_i^\dagger |N_1, N_2, \dots, N_i, \dots\rangle \Rightarrow |N_1, N_2, \dots, N_i + 1, \dots\rangle. \quad (2.13)$$

The operators \hat{a}_i and \hat{a}_i^\dagger are called annihilation and creation operators of particles. In order to write the operator $\hat{F}^{(1)}$ as

$$\hat{F}^{(1)} = \sum_{i,k} f_{ik}^{(1)} \hat{a}_i^\dagger \hat{a}_k, \quad (2.14)$$

like in the case of bosons, the annihilation and creation operators should be determined as matrices with elements

$$\langle 0_i | \hat{a}_i | 1_i \rangle = \langle 1_i | \hat{a}_i^\dagger | 0_i \rangle = (-1)^{A_{0i}}. \quad (2.15)$$

For the product of such matrices we find (for $i < k$):

$$\langle 1_i, 0_k | \hat{a}_i^\dagger \hat{a}_k | 0_i, 1_k \rangle = \langle 1_i, 0_k | \hat{a}_i^\dagger | 0_i, 0_k \rangle \times \langle 0_i, 0_k | \hat{a}_k | 0_i, 1_k \rangle = (-1)^{[A_{0i} + A_{0k}]}. \quad (2.16)$$

Since in the second matrix element the occupation number $N_i = 0$, we may write

$$A_{0k} = \sum_{j=1}^{k-1} N_j = \sum_{j=1}^{i-1} N_j + \sum_{j=i+1}^{k-1} N_j = A_{0i} + A_{ik}.$$

We then obtain

$$\langle 1_i, 0_k | \hat{a}_i^\dagger \hat{a}_k | 0_i, 1_k \rangle = (-1)^{A_{ik}}. \quad (2.17)$$

For $i = k$ the matrix $\hat{a}_i^\dagger \hat{a}_i$ is diagonal:

$$\hat{a}_i^\dagger \hat{a}_i = N_i \quad (2.18)$$

(zero element for $N_i = 0$ and unity for $N_i = 1$).

Reversing the order of the operators we again obtain the result of Eq. (2.16) ($i < k$):

$$\langle 1_i, 0_k | \hat{a}_k \hat{a}_i^\dagger | 0_i, 1_k \rangle = \langle 1_i, 0_k | \hat{a}_k | 1_i, 0_k \rangle \times \langle 1_i, 1_k | \hat{a}_i^\dagger | 0_i, 1_k \rangle = (-1)^{[A_{0i} + A_{0k}]}.$$

However, now in the first matrix element the occupation number $N_i = 1$ and, hence,

$$A_{0i} = \sum_{j=1}^{i-1} N_j = \sum_{j=1}^{k-1} N_j - \sum_{j=i}^{k-1} N_j = A_{0k} - A_{ik} - 1.$$

Thus, we get

$$\langle 1_i, 0_k | \hat{a}_k \hat{a}_i^\dagger | 0_i, 1_k \rangle = (-1)^{2A_{0k} - A_{ik} - 1} = (-1) \times (-1)^{A_{ik}} \quad (2.19)$$

So, comparing Eq. (2.19) with Eq. (2.17) we see that

$$\hat{a}_i^\dagger \hat{a}_k + \hat{a}_k \hat{a}_i^\dagger = 0; \quad i \neq k. \quad (2.20)$$

For the diagonal matrix we can write:

$$\hat{a}_i \hat{a}_i^\dagger = 1 - N_i, \quad (2.21)$$

so that

$$\hat{a}_i \hat{a}_i^\dagger + \hat{a}_i^\dagger \hat{a}_i = 1. \quad (2.22)$$

A general relation can be written as

$$\hat{a}_i \hat{a}_k^\dagger + \hat{a}_k^\dagger \hat{a}_i = \delta_{ik}. \quad (2.23)$$

Similarly, we can obtain

$$\hat{a}_i \hat{a}_k + \hat{a}_k \hat{a}_i = 0. \quad (2.24)$$

The Hamiltonian (2.1) has only diagonal matrix elements. In free space the particle momentum is a good quantum number and plane waves form a complete set of basis functions. Thus, in terms of the operators $\hat{a}_{\mathbf{k}}$ and $\hat{a}_{\mathbf{k}}^\dagger$, equation (2.1) can be written as

$$\hat{H}_{kin} = \sum_{\mathbf{k}} E_k \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}}, \quad (2.25)$$

with $E_k = \hbar^2 k^2 / 2m$ being the single particle kinetic energy.

Relying on Eqs. (2.23) and (2.24) we can establish commutation relations for the field operators $\hat{\psi}(\mathbf{r})$ and $\hat{\psi}^\dagger(\mathbf{r})$ defined as

$$\hat{\psi}(\mathbf{r}) = \sum_i \hat{a}_i \psi_i(\mathbf{r}), \quad (2.26)$$

$$\hat{\psi}^\dagger(\mathbf{r}) = \sum_i \hat{a}_i^\dagger \psi_i^*(\mathbf{r}), \quad (2.27)$$

and representing annihilation and creation operators of a fermionic particle at the point \mathbf{r} . We have

$$\hat{\psi}^\dagger(\mathbf{r})\hat{\psi}(\mathbf{r}') + \hat{\psi}(\mathbf{r}')\hat{\psi}^\dagger(\mathbf{r}) = \delta(\mathbf{r} - \mathbf{r}'). \quad (2.28)$$

$$\hat{\psi}(\mathbf{r})\hat{\psi}(\mathbf{r}') + \hat{\psi}(\mathbf{r}')\hat{\psi}(\mathbf{r}) = 0. \quad (2.29)$$

In terms of the field operators $\hat{\psi}(\mathbf{r})$, $\hat{\psi}^\dagger(\mathbf{r})$ the Hamiltonian (2.25) takes the form:

$$\hat{H}_{kin} = - \int d^3r \hat{\psi}^\dagger(\mathbf{r}) \frac{\hbar^2}{2m} \Delta_{\mathbf{r}} \hat{\psi}(\mathbf{r}). \quad (2.30)$$

We thus see that the fermionic operators anticommute, in contrast to bosonic operators which commute. In a mixture of bosons and fermions, the operators of the fermions commute with bosonic operators. In a multicomponent Fermi system, the operators of different fermions can be considered as both commutative or anticommutative. The final results for physical quantities remain the same.

We finally present a useful relation for the square of the matrix element of the operator $\hat{a}_i^\dagger \hat{a}_k$ for $i \neq k$:

$$|\langle N_i + 1, N_k - 1 | \hat{a}_i^\dagger \hat{a}_k | N_i, N_k \rangle|^2 = \langle N_i, N_k | \hat{a}_i \hat{a}_i^\dagger \hat{a}_k^\dagger \hat{a}_k | N_i, N_k \rangle = N_k(1 - N_i). \quad (2.31)$$

Let us now consider a two-component Fermi gas and omit the interaction between identical fermions. Then we only have the s -wave interaction between fermions of different components (internal states), which we denote as \uparrow and \downarrow . In the first quantization the interaction Hamiltonian reads:

$$\hat{H}_{int} = \sum_{\alpha\uparrow, \beta\downarrow} U(\mathbf{r}_{\alpha\uparrow} - \mathbf{r}_{\beta\downarrow}), \quad (2.32)$$

where $\mathbf{r}_{\alpha\uparrow}$ and $\mathbf{r}_{\beta\downarrow}$ are coordinates of \uparrow and \downarrow fermions, and $U(\mathbf{r})$ is the potential of pair interaction between such fermions. The amplitude of the s -wave scattering in the ultracold limit is momentum independent and equal to $-a$, where a is the scattering length. The corresponding coupling constant for the interparticle interaction is $g = 4\pi\hbar^2 a/m$. Then, the secondly quantized interaction Hamiltonian in terms of the field operators $\hat{\psi}(\mathbf{r})$, $\hat{\psi}^\dagger(\mathbf{r})$ takes the form:

$$\hat{H}_{int} = g \int d^3r \hat{\psi}_\uparrow^\dagger(\mathbf{r}) \hat{\psi}_\downarrow^\dagger(\mathbf{r}) \hat{\psi}_\downarrow(\mathbf{r}) \hat{\psi}_\uparrow(\mathbf{r}). \quad (2.33)$$

In free space, where the wavevector \mathbf{k} of a particle is a good quantum number, equation (2.33) can be rewritten through the creation and annihilation operators in the momentum space:

$$\hat{H}_{int} = \frac{g}{V} \sum_{\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3} \hat{a}_{\mathbf{k}_3\uparrow}^\dagger \hat{a}_{\mathbf{k}_4\downarrow}^\dagger \hat{a}_{\mathbf{k}_1\downarrow} \hat{a}_{\mathbf{k}_2\uparrow}, \quad (2.34)$$

and there is a momentum conservation law $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 + \mathbf{k}_4$. The derivation of equations (2.34) and (2.33) is exactly the same as in the case of bosons in subsections 2.2 and 2.3 of *Lecture 2* in *Part 1* of the course and we do not repeat this derivation here.

2.2 Thermodynamics of an ideal Fermi gas

We now turn to thermodynamic properties of an ideal Fermi gas. For simplicity, we first consider a single-component Fermi gas in free space. It is described by the Hamiltonian (2.25). As we discussed in subsection 1.1 of *Lecture 1*, the distribution function (occupation number averaged over the ensemble) at $T = 0$ is a filled Fermi sphere: $N_{\mathbf{k}} = \theta(k_F - k)$, where the Fermi wavevector k_F and Fermi energy E_F are given by Eqs. (1.5) and (1.6), and $\theta(x)$ is the step function equal to zero for negative arguments and to unity for positive ones. So, all states with $E < E_F$ or $k < k_F$ are occupied, and the states with higher energies (above the Fermi surface) are empty. The zero-temperature distribution function $N_{\mathbf{k}}$ is given in black in Fig. 1.1. It is the limiting case of the general Fermi distribution (1.1) at $T \rightarrow 0$, and the zero-temperature chemical potential is $\mu_0 = E_F$ (see subsection 1.1 of *Lecture 1*). The energy of the ideal single-component Fermi gas at $T = 0$ is

$$E = \int N_{\mathbf{k}} E_{\mathbf{k}} \frac{V d^3 k}{(2\pi)^3} = \int_0^{k_F} \frac{\hbar^2}{m} k^4 \frac{V dk}{4\pi^2} = \frac{\hbar^2 k_F^5 V}{20\pi^2 m} = \frac{\hbar^2 V}{20\pi^2 m} \left(\frac{6\pi^2 N}{V} \right)^{5/3} = \frac{3}{5} E_F N, \quad (2.35)$$

where we used equation (1.4) providing a relation between the Fermi momentum $\hbar k_F$ and the total number of particles N .

Let us now consider temperatures $T \ll E_F$ and develop calculations of thermodynamic functions of the ideal Fermi gas. At these temperatures the chemical potential is close to the Fermi energy and our first step will be to find the deviation of μ from E_F . For this purpose we turn from the integration over k to the integration over $E_k = \hbar^2 k^2 / 2m$ in the expression providing a relation between the density and chemical potential through the Fermi distribution function. Using Eq. (1.1) we have:

$$n = \int \frac{d^3 k}{(2\pi)^3} N_{\mathbf{k}} = \int_0^\infty \frac{4\pi k^2 dk}{(2\pi)^3} \frac{1}{\exp\{(E_k - \mu)/T\} + 1} = \int_0^\infty \frac{\nu(E) dE}{\exp\{(E - \mu)/T\} + 1}, \quad (2.36)$$

where

$$\nu(E) = \left(\frac{2m}{\hbar^2} \right)^{3/2} \frac{E^{1/2}}{4\pi^2} \quad (2.37)$$

is the density of states. We then represent Eq. (2.36) as

$$n = \int_{-\mu}^0 \frac{\nu(E - \mu + \mu) d(E - \mu)}{\exp\{(E - \mu)/T\} + 1} + \int_0^\infty \frac{\nu(E - \mu + \mu) d(E - \mu)}{\exp\{(E - \mu)/T\} + 1}$$

and write in the first integral

$$\frac{1}{\exp\{(E - \mu)/T\} + 1} = 1 - \frac{\exp\{(E - \mu)/T\}}{\exp\{(E - \mu)/T\} + 1} = 1 - \frac{1}{\exp(\epsilon/T) + 1},$$

where $\epsilon = \mu - E$. In the second integral we write $E - \mu = \epsilon$. Then we have:

$$n = - \int_0^\mu \frac{\nu(\mu - \epsilon) d\epsilon}{\exp(\epsilon/T) + 1} + \int_0^\infty \frac{\nu(\mu + \epsilon) d\epsilon}{\exp(\epsilon/T) + 1} + \int_0^\mu \nu(E) dE. \quad (2.38)$$

The values of ϵ which give the main contribution to the first two integrals are of the order of $T \ll \mu$. Therefore, we extend the integration to infinity in the first integral, and then we expand the density of states $\nu(\mu \pm \epsilon)$ in powers of ϵ up to the linear term:

$$\nu(\mu \pm \epsilon) = \nu(\mu) \pm \epsilon \frac{d\nu}{dE}.$$

This yields

$$n = \int_0^\mu \nu(E) dE + 2T^2 \frac{d\nu(\mu)}{d\mu} \int_0^\infty \frac{xdx}{\exp(x) + 1}.$$

Using Eq. (2.37) and the fact that $\int_0^\infty x[\exp(x) + 1]^{-1} dx = \pi^2/12$ we then find

$$n = \left(\frac{1}{6\pi^2} + \frac{T^2}{48\mu^2} \right) \left(\frac{2m\mu}{\hbar^2} \right)^{3/2}. \quad (2.39)$$

The second term in Eq. (2.39) is much smaller than the first one and we may put $\mu = E_F$ in this term, whereas in the first term we should write $\mu = E_F + \delta$, where δ is a small deviation of the chemical potential from the Fermi energy. We thus obtain a linear equation for δ , which gives

$$\delta = -\frac{\pi^2}{12} \left(\frac{T}{E_F} \right)^2 E_F. \quad (2.40)$$

In a similar way we calculate the energy of the gas:

$$\begin{aligned} E &= \int \frac{V d^3k}{(2\pi)^3} \frac{E_k}{\exp\{(E_k - \mu)/T\} + 1} = V \int_0^\infty \frac{E\nu(E)dE}{\exp\{(E - \mu)/T\} + 1} \\ &= V \int_0^\mu E\nu(E)dE \left[1 - \frac{\exp\{(E - \mu)/T\}}{\exp\{(E - \mu)/T\} + 1} \right] + V \int_0^\infty \frac{E\nu(E)d(E - \mu)}{\exp\{(E - \mu)/T\} + 1} \\ &= V \int_0^\mu E\nu(E)dE + 2V \frac{d(E\nu(E))}{dE} \Big|_{E=\mu} \times \int_0^\infty \frac{(E - \mu)d(E - \mu)}{\exp\{(E - \mu)/T\} + 1} \\ &= \frac{3}{5} E_F N \left(\frac{\mu}{E_F} \right)^{5/2} + \frac{\pi^2}{4} \nu(\mu) T^2 V = \frac{3}{5} E_F N + \frac{3}{2} \delta + \frac{3\pi^2}{8} \left(\frac{T}{E_F} \right)^2 E_F N \\ &= \frac{3}{5} E_F N + \frac{\pi^2}{4} \frac{T^2}{E_F} N. \end{aligned} \quad (2.41)$$

The free energy F is related to the energy E by

$$E = -T^2 \left(\frac{\partial F}{\partial T} \right)_V \quad (2.42)$$

and the entropy is given by the relation

$$S = - \left(\frac{\partial F}{\partial T} \right)_V. \quad (2.43)$$

The relation for the pressure is

$$P = - \left(\frac{\partial F}{\partial V} \right)_T. \quad (2.44)$$

These thermodynamic quantities are easily calculated using Eq. (2.41) for the energy E .

The generalization to multicomponent ideal Fermi gases is straightforward. In the absence of interactions, fermionic components behave themselves as independent ideal gases. For example, in the case of two components which we again denote as \uparrow and \downarrow , the distribution of particles in these components is given by

$$N_{\mathbf{p}\uparrow} = \frac{1}{\exp\{(E_{\mathbf{k}\uparrow} - \mu_{\uparrow})/T\} + 1}, \quad (2.45)$$

$$N_{\mathbf{p}\downarrow} = \frac{1}{\exp\{(E_{\mathbf{k}\downarrow} - \mu_{\downarrow})/T\} + 1}, \quad (2.46)$$

where the corresponding chemical potentials are $\mu_{\uparrow} = E_{F\uparrow} + \delta_{\uparrow}$ and $\mu_{\downarrow} = E_{F\downarrow} + \delta_{\downarrow}$. The quantities δ_{\uparrow} and δ_{\downarrow} are given by Eq. (2.40) with $E_F = E_{F\uparrow}$ and $E_F = E_{F\downarrow}$, respectively. The Fermi energies of the two components, $E_{F\uparrow}$ and $E_{F\downarrow}$ follow from Eq. (1.6). One should simply put $n = n_{\uparrow} = N_{\uparrow}/V$ for the \uparrow -component, and $n = n_{\downarrow} = N_{\downarrow}/V$ for the \downarrow -component. The energy of the two-component gas is $E = E_{\uparrow} + E_{\downarrow}$, where E_{\uparrow} and E_{\downarrow} are given by Eq. (2.41) with $E_F = E_{F\uparrow}$, $N = N_{\uparrow}$ and $E_F = E_{F\downarrow}$, $N = N_{\downarrow}$, respectively.

2.3 Particle and hole excitations

We now consider $T = 0$ and discuss excitations of an ideal Fermi gas in free space. The ground state is a filled Fermi sphere, and let us create an excited state by transferring a particle with momentum $\hbar\mathbf{k}_1$ ($|\mathbf{k}_1| < k_F$) to the state with momentum $\hbar\mathbf{k}_2$ ($|\mathbf{k}_2| > k_F$) as depicted in Fig. 2.2. The change of the energy is

$$E - E_0 = \frac{\hbar^2 k_2^2}{2m} - \frac{\hbar^2 k_1^2}{2m},$$

and it can be written in the form

$$E - E_0 = \left(\frac{\hbar^2 k_2^2}{2m} - E_F \right) + \left(E_F - \frac{\hbar^2 k_1^2}{2m} \right). \quad (2.47)$$

By taking a particle out of the Fermi sphere and putting it into an excited state above the Fermi surface we create a hole in the Fermi sphere. Thus, we can distinguish between particle and hole excitations.

Consider first *particle excitations*. Acting with the operator $\hat{a}_{\mathbf{k}}^{\dagger}$ on the ground state of N particles we have $\hat{a}_{\mathbf{k}}^{\dagger}|N\rangle \neq 0$ only for $k > k_F$. In the latter case we create an excited state of the system of $(N + 1)$ particles. The excitation energy is given by

$$\epsilon_k = E_k + E_0(N) - E_0(N + 1) = \frac{\hbar^2 k^2}{2m} - E_F > 0; \quad k > k_F, \quad (2.48)$$

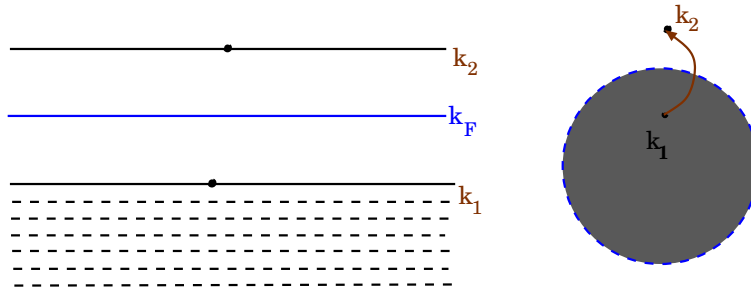


Figure 2.2: The creation of particle and hole excitations.

i.e. it is represented by the first term in the right hand side of equation (2.47). So, for $k > k_F$ the quantity ϵ_k (2.48) can be identified as a particle excitation.

Let us now consider *hole excitations*. Acting with the operator $\hat{a}_{\mathbf{k}}$ on the ground state of N particles we have $\hat{a}_{\mathbf{k}}|N\rangle \neq 0$ only for $k < k_F$. Then, if $k < k_F$, we create an excited state of $(N - 1)$ particles. The corresponding excitation energy is

$$\epsilon_k = E_0(N) - E_0(N - 1) - E_k = E_F - \frac{\hbar^2 k^2}{2m} > 0; \quad k < k_F, \quad (2.49)$$

and is equivalent to the second term in the right hand side of Eq. (2.47). Thus, for $k < k_F$ the quantity ϵ_k (2.49) can be treated as a hole excitation. The particle and hole excitation branches are displayed in Fig. 2.3. In both cases one can write the excitation energy as

$$\epsilon_k = \frac{\hbar^2 |k^2 - k_F^2|}{2m}. \quad (2.50)$$

Problems 2

2.1 Calculate the chemical potential and total energy of a single-component ideal two-dimensional Fermi gas at a finite temperature.

We first introduce the density of states in two dimensions (2D):

$$\nu(E) = \int \frac{d^2 k}{(2\pi)^2} \delta(E - \hbar^2 k^2 / 2m) = \frac{m}{2\pi \hbar^2} = \text{const.} \quad (2.51)$$

Then, we obtain the following relation between the chemical potential and density:

$$n = \int_0^\infty \frac{\nu(E) dE}{\exp\{(E - \mu)/T\} + 1} = \frac{mT}{2\pi \hbar^2} \ln [1 + \exp(\mu/T)]. \quad (2.52)$$

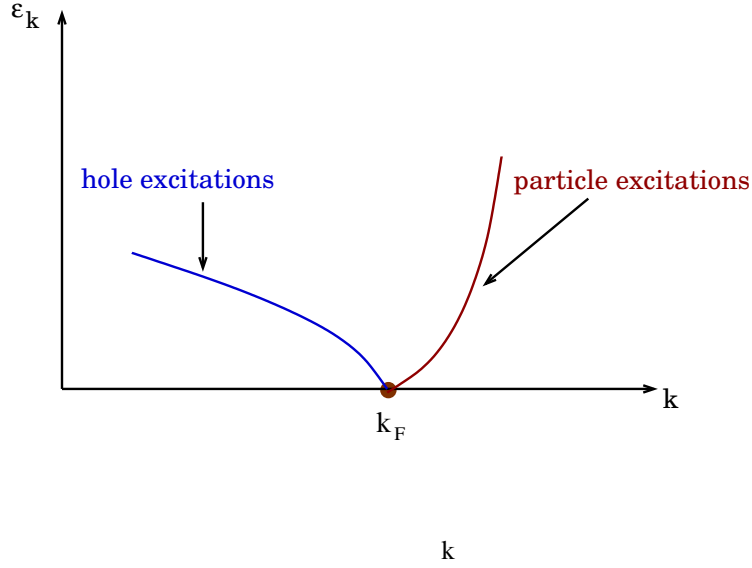


Figure 2.3: The energies of particle and hole excitations versus the momentum $\hbar k$.

We thus obtain:

$$\mu = T \ln [\exp(T_d/T) - 1], \quad (2.53)$$

where the temperature of quantum degeneracy is given by

$$T_d = \frac{2\pi\hbar^2}{m} n \quad (2.54)$$

and is equal to the 2D Fermi energy E_F .

At high temperatures, $T \gg T_d$, equation (2.53) gives the well-known classical result:

$$\mu = -T \ln \left(\frac{T}{T_d} \right) = T \ln(n\Lambda_T^2). \quad (2.55)$$

At low temperatures, $T \ll T_d = E_F$, we have

$$\mu = E_F - T \exp(-E_F/T). \quad (2.56)$$

We thus see that the temperature-dependent part of μ is exponential, in contrast to the 3D case where it is proportional to T^2 .

For the total energy at $T \ll E_F$ we find

$$\begin{aligned} E &= V \int_0^\infty \frac{\nu(E) E dE}{\exp\{(E - \mu)/T\} + 1} = \frac{mV\mu^2}{4\pi\hbar^2} + \frac{\pi T^2 mV}{12\hbar^2} \\ &= \frac{1}{2} E_F N + \frac{\pi^2}{6} \left(\frac{T}{E_F} \right)^2 E_F N, \end{aligned} \quad (2.57)$$

where we used the same method of integration as in 3D and omitted exponentially small corrections to μ .

2.2 Calculate the chemical potential and total energy for the one-dimensional ideal Fermi gas at temperatures $T \ll E_F$.

Lecture 3. Repulsively interacting Fermi gas. Landau's Fermi liquid theory

3.1 Weakly interacting Fermi gas with repulsion between particles

We now consider a degenerate interacting two-component Fermi system and start with the weakly interacting Fermi gas. We only include the s -wave interaction between fermions of different components and in this *Lecture* discuss the case where the interaction is repulsive. So, in free space the Hamiltonian is the sum of the kinetic and interaction energy terms which in the momentum space second quantization are given by Eqs. (2.25) and (2.34), respectively:

$$\hat{H} = \sum_{\mathbf{k}} \frac{\hbar^2 k^2}{2m} (\hat{a}_{\uparrow\mathbf{k}}^\dagger \hat{a}_{\uparrow\mathbf{k}} + \hat{a}_{\downarrow\mathbf{k}}^\dagger \hat{a}_{\downarrow\mathbf{k}}) + \frac{g}{V} \sum_{\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4} \hat{a}_{\uparrow\mathbf{k}_3}^\dagger \hat{a}_{\downarrow\mathbf{k}_4}^\dagger \hat{a}_{\downarrow\mathbf{k}_2} \hat{a}_{\uparrow\mathbf{k}_1}, \quad (3.1)$$

where $g = 4\pi\hbar^2 a/m$ with $a > 0$ being the scattering length, the symbols \uparrow and \downarrow label the two fermionic components, and the momentum conservation law $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 + \mathbf{k}_4$ is satisfied. As we discussed in subsection 1.4 of *Lecture 1*, the weakly interacting regime is characterized by the condition $na^3 \ll 1$ which ensures that the interaction energy is much smaller than the Fermi energy (kinetic energy of particles) and allows one to treat the interaction part of the Hamiltonian within a many-body perturbation theory. To zero order the total energy of the system at $T = 0$ is equal to the ideal gas energy $E_0 = (3/5)[E_{F\uparrow}N_\uparrow + E_{F\downarrow}N_\downarrow]$, where $E_{F\uparrow}, N_\uparrow$ and $E_{F\downarrow}, N_\downarrow$ are Fermi energies and particle numbers in the \uparrow and \downarrow components (see subsection 2.2 of *Lecture 2*).

In the first order of perturbation theory, we have to add the diagonal matrix element of the interaction term (the second term of Eq. (3.1)). It is easy to see that then we should have $\mathbf{k}_3 = \mathbf{k}_1$ and $\mathbf{k}_4 = \mathbf{k}_2$, otherwise the diagonal matrix element is zero. So, the contribution of the interaction term to the total energy is

$$E_{int} = \frac{g}{V} \sum_{\mathbf{k}_1, \mathbf{k}_2} \langle \hat{a}_{\uparrow\mathbf{k}_1}^\dagger \hat{a}_{\uparrow\mathbf{k}_1} \rangle \langle \hat{a}_{\downarrow\mathbf{k}_2}^\dagger \hat{a}_{\downarrow\mathbf{k}_2} \rangle = \frac{g}{V} \sum_{\mathbf{k}_1, \mathbf{k}_2} N_{\uparrow\mathbf{k}_1} N_{\downarrow\mathbf{k}_2} = \frac{g}{V} N_\uparrow N_\downarrow. \quad (3.2)$$

Thus, the total energy of the Fermi gas is given by

$$E_0 = \frac{3}{5}(E_{F\uparrow}N_\uparrow + E_{F\downarrow}N_\downarrow) + \frac{g}{V}N_\uparrow N_\downarrow, \quad (3.3)$$

and for the chemical potentials of the \uparrow and \downarrow components we have:

$$\mu_{\uparrow} = \frac{\partial E_0}{\partial N_{\uparrow}} = E_{F\uparrow} + n_{\downarrow}g, \quad (3.4)$$

$$\mu_{\downarrow} = \frac{\partial E_0}{\partial N_{\downarrow}} = E_{F\downarrow} + n_{\uparrow}g, \quad (3.5)$$

If the concentrations of the two fermionic components are equal, i.e. $N_{\uparrow} = N_{\downarrow} = N/2$ and $E_{F\uparrow} = E_{F\downarrow} = E_F$, then equations (3.3), (3.4), and (3.5) yield

$$E_0 = \left(\frac{3}{5}E_F + \frac{1}{4}ng \right) N = \frac{3}{5}E_F N \left(1 + \frac{5}{12} \frac{ng}{E_F} \right) = \frac{3}{5}E_F N \left(1 + \frac{10}{9\pi} k_F a \right), \quad (3.6)$$

$$\mu_{\uparrow} = \mu_{\downarrow} = \mu = E_F + \frac{1}{2}ng = E_F \left(1 + \frac{ng}{2E_F} \right) = E_F \left(1 + \frac{4}{3\pi} k_F a \right). \quad (3.7)$$

The small parameter of the perturbation theory is $(ng/E_F) \sim k_F a \ll 1$. As we see, the first order correction to the energy is $\sim k_F a$. The second order correction turns out to be $\sim (k_F a)^2$. The calculation of this correction is beyond the scope of the present course.

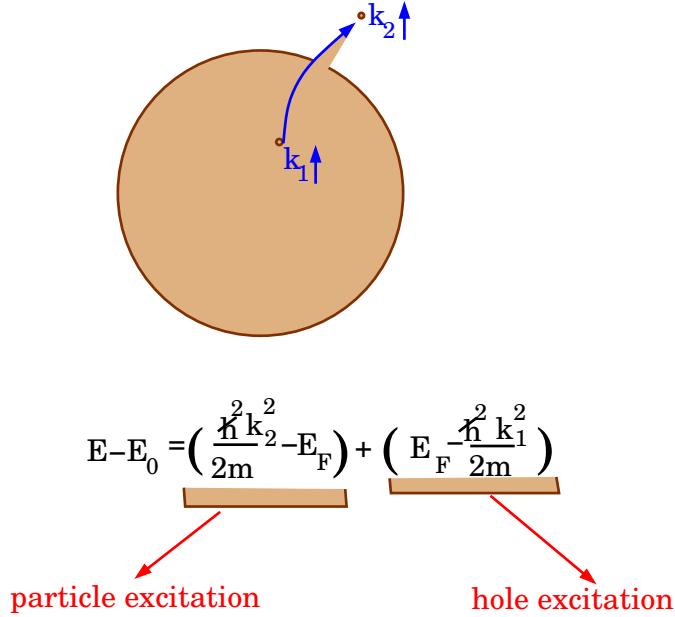


Figure 3.1: Particle and hole excitations in a two-component Fermi gas.

Let us now discuss excitations of a weakly interacting Fermi gas with interparticle repulsion. The picture of particle and hole excitations in the two-component gas remains the same as discussed in subsection 2.3 for a single-component gas. For $T = 0$, transferring for example a \uparrow particle from the

state with wavevector \mathbf{k}_1 inside the Fermi sphere to the state with wavevector \mathbf{k}_2 above the Fermi surface we change the energy by an amount $E - E_0 = (\hbar^2 k_2^2/2m - \hbar^2 k_1^2/2m)$ and create both a particle and a hole excitation as shown in Fig.3.1. The particle excitation is an excited state of the system of $(N + 1)$ particles. By bringing an extra \uparrow -particle from infinity and putting it above the Fermi surface we increase the energy by $(\hbar^2 k_2^2/2m - E_F)$ compared to the ground state of $(N + 1)$ particles. The interaction energy is the same as its momentum independent. The hole excitation is an excited state of $(N - 1)$ particles. Making a hole we increase the energy by $(E_F - \hbar^2 k_1^2/2m)$ compared to the ground state of $(N - 1)$ particles. Again, the interaction energy is the same. So, the excitation branches behave themselves exactly in the same way as depicted in Fig.2.2 for the ideal gas.

Does the interaction do something with the excitations? The answer is yes, it does. In contrast to the ideal gas, the excitations of an interacting Fermi gas interact with each other and have a finite lifetime, i.e. they are damped. Consider a particle excitation with momentum $\hbar\mathbf{k}_1$ ($k_1 > k_F$) at $T = 0$. It interacts with a particle which has momentum $\hbar\mathbf{k}_2$ and is located inside the Fermi sphere ($k_2 < k_F$), makes a hole and creates two particle excitations, with momenta $\hbar\mathbf{k}'_1$ and $\hbar\mathbf{k}'_2$. Using the terminology of the scattering theory, \uparrow and \downarrow particles with momenta $\hbar\mathbf{k}_1$ and $\hbar\mathbf{k}_2$, collide and go to the states with momenta $\hbar\mathbf{k}'_1$ and $\hbar\mathbf{k}'_2$. In the language of particle and hole excitations, a particle excitation with wavevector \mathbf{k}_1 (close to k_F) and energy $E = \hbar^2 k_1^2/2m - E_F$ decays into two particle excitations and one hole excitation. This process is caused by the interaction Hamiltonian (2.34):

$$\frac{g}{V} \sum_{\mathbf{k}_2, \mathbf{k}'_1} \hat{a}_{\uparrow\mathbf{k}'_1}^\dagger \hat{a}_{\downarrow\mathbf{k}'_2}^\dagger \hat{a}_{\downarrow\mathbf{k}_2} \hat{a}_{\uparrow\mathbf{k}_1}.$$

The decay rate is obtained by using the Fermi Golden rule:

$$\frac{1}{\tau} = \sum_{\mathbf{k}_2, \mathbf{k}'_1} \frac{2\pi}{\hbar} \left| \langle \uparrow \mathbf{k}'_1, \downarrow \mathbf{k}'_2 | \hat{H}_{int} | \downarrow \mathbf{k}_2, \uparrow \mathbf{k}_1 \rangle \right|^2 \delta \left(\frac{\hbar^2 k_1^2}{2m} + \frac{\hbar^2 k_2^2}{2m} - \frac{\hbar^2 k_1'^2}{2m} - \frac{\hbar^2 k_2'^2}{2m} \right), \quad (3.8)$$

and there is a momentum conservation law $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}'_1 + \mathbf{k}'_2$.

We now write $k_2'^2 = |\mathbf{k}_1 + \mathbf{k}_2|^2 + k_1'^2 - 2k_1'|\mathbf{k}_1 + \mathbf{k}_2| \cos \theta'$, where θ' is the angle between the vectors $(\mathbf{k}_1 + \mathbf{k}_2)$ and \mathbf{k}'_1 . Since all momenta are close to $\hbar k_F$, we may put $k_1 = k_2 = k_1' = k_F$ and set $|\mathbf{k}_1 + \mathbf{k}_2| = 2k_F \cos \theta/2$ (θ is the angle between \mathbf{k}_1 and \mathbf{k}_2) in the energy conservation law, i.e. in the argument of the δ -function in Eq. (3.8). Then this argument becomes equal to $(2\hbar^2 k_F^2/m)[\cos^2 \theta/2 - \cos \theta/2 \cos \theta']$, and we rewrite Eq. (3.8) as

$$\begin{aligned} \frac{1}{\tau} &= \frac{2\pi g^2}{\hbar} \int \frac{2\pi k_1'^2 dk_1' 2\pi k_2^2 dk_2 \sin \theta d\theta d \cos \theta'}{(2\pi)^6} N_{\downarrow}(k_2)(1 - N_{\uparrow}(k_1'))(1 - N_{\downarrow}(k_2')) \\ &\times \delta \left(\frac{2\hbar^2 k_F^2}{m} [\cos^2 \theta/2 - \cos \theta/2 \cos \theta'] \right), \end{aligned} \quad (3.9)$$

where the occupation numbers $N_{\uparrow}(k)$ and $N_{\downarrow}(k)$ are equal to unity for $k < k_F$ and to zero for $k > k_F$, and $k'_2 = \sqrt{k_1^2 + k_2^2 - k_1'^2}$. The integration over $d \cos \theta'$ and then over $d\theta$ gives

$$\frac{1}{\tau} = \frac{2mg^2k_F^2}{(2\pi\hbar)^3} \int dq'_1 dq_2 N_{\downarrow}(k_F + q_2)(1 - N_{\uparrow}(k_F + q'_1))(1 - N_{\downarrow}(k_F + q_1 + q_2 - q'_1)). \quad (3.10)$$

Here $q_1 = k_1 - k_F$, $q_2 = k_2 - k_F$, $q'_1 = k'_1 - k_F$, and hence $q'_2 = q_1 + q_2 - q'_1$. In order to have $N_{\downarrow}(k_F + q_2) = 1$, $N_{\uparrow}(k_F + q'_1) = 0$, $N_{\downarrow}(k_F + q_1 + q_2 - q'_1) = 0$, the momenta should satisfy the inequalities

$$q_2 < 0; \quad q'_1 > 0; \quad q_1 + q_2 - q'_1 > 0.$$

Since we consider the decay of a particle excitation we have $q_1 > 0$, and these inequalities are equivalent to

$$0 < q'_1 < q_1 + q_2, \quad (3.11)$$

$$-q_1 < q_2 < 0. \quad (3.12)$$

Equations (3.11) and (3.12) determine the limits of integration over dq'_1 and dq_2 in Eq. (3.10). We then obtain:

$$\frac{1}{\tau} = \frac{mk_F^2 g^2 q_1^2}{(2\pi\hbar)^3}. \quad (3.13)$$

As we consider the decay of an excitation with momentum k_1 close to the Fermi surface, the excitation energy can be written in the form $E = \hbar^2 k_1^2 / 2m - E_F = \hbar v_F q_1$, where $v_F = \hbar k_F / m$ is the Fermi velocity. Recalling that $k_F^3 = 3\pi^2 n$ and $g = 4\pi\hbar^2 a / m$ we then rewrite equation (3.13) as

$$\frac{1}{\tau} = \frac{3}{8} n \sigma v_F \left(\frac{E}{E_F} \right)^2, \quad (3.14)$$

where $\sigma = 4\pi a^2$ is the cross section of elastic collisions. We thus see that the presence of the filled Fermi sphere (Pauli blocking) makes the lifetime of excitations rather large. It is clear that

$$\frac{1}{\tau} \ll \frac{E}{\hbar}. \quad (3.15)$$

The ratio $E\tau/\hbar$ is

$$\frac{E\tau}{\hbar} \simeq \pi \left(\frac{E_F}{E} \right) \frac{1}{(k_F a)^2} \gg 1. \quad (3.16)$$

3.2 Quasiparticles in Landau's Fermi liquid theory

We now turn to the discussion of Landau's Fermi liquid theory which was designed for the description of strongly (repulsively) interacting Fermi systems,

such as liquid ${}^3\text{He}$. In a strongly interacting Fermi system instead of considering particles one introduces dressed particles, or *quasiparticles*. The number of quasiparticles is equal to the total number of particles and they also obey Fermi statistics. In free space each quasiparticle has momentum \mathbf{p} and energy $\epsilon(\mathbf{p})$ (in the rest of this *Lecture* \mathbf{p} is the true momentum, not the wavevector). Let $n(\mathbf{p})$ be the distribution function of quasiparticles. Then, in a two-component Fermi liquid, assuming that $\epsilon(\mathbf{p})$ and $n(\mathbf{p})$ are spin independent, we have

$$2 \int n(\mathbf{p}) \frac{d^3p}{(2\pi\hbar)^3} = \frac{N}{V}, \quad (3.17)$$

where N is the total number of particles. However, the total energy E is *not equal* to $2 \int n(\mathbf{p}) \epsilon(\mathbf{p}) V d^3p / (2\pi\hbar)^3$. The energy E is a functional of $n(\mathbf{p})$. Considering a change of E under an infinitesimally small variation δn of $n(\mathbf{p})$ we have

$$\frac{\delta E}{V} = 2 \int \epsilon(\mathbf{p}) \delta n(\mathbf{p}) \frac{d^3p}{(2\pi\hbar)^3}. \quad (3.18)$$

So, $\epsilon(\mathbf{p})$ is a variational derivative of E with respect to $n(\mathbf{p})$.

The classification of energy levels in a Fermi liquid is similar to that in an ideal Fermi gas. The only difference is that particles are replaced by quasiparticles. Therefore, the entropy is given by the well-known expression:

$$\frac{S}{V} = -2 \int \frac{d^3p}{(2\pi\hbar)^3} \{n(\mathbf{p}) \ln n(\mathbf{p}) + (1 - n(\mathbf{p})) \ln (1 - n(\mathbf{p}))\}, \quad (3.19)$$

and the distribution of quasiparticles at equilibrium obeys the Fermi-Dirac relation:

$$n(\mathbf{p}) = \frac{1}{\exp\{[\epsilon(\mathbf{p}) - \mu]/T\} + 1}. \quad (3.20)$$

At $T = 0$ the chemical potential coincides with the boundary energy at the Fermi sphere:

$$\mu_0 \equiv \mu(T = 0) = \epsilon_F \equiv \epsilon(p_F). \quad (3.21)$$

Note, however, that $\epsilon(\mathbf{p})$ itself may depend on $n(\mathbf{p})$. The distribution of quasiparticles at $T = 0$ represents a step function:

$$n(\mathbf{p}) = \theta(p_F - p) \equiv \begin{cases} 1, & p < p_F \\ 0, & p > p_F \end{cases} \quad (3.22)$$

and the Fermi momentum is related to the total density n by the same expression as in an ideal Fermi gas: $p_F = \hbar(3\pi^2 n)^{1/3}$. At finite low temperatures the distribution function is different from the step function (3.22) in the energy interval $\sim T$ near the boundary energy ϵ_F .

The fact that a quasiparticle has momentum and energy means, in particular, that the uncertainty in the energy should be smaller than the least of the important energy scales in the system. As we saw in previous lectures of the course, in a weakly interacting Fermi gas at temperatures $T \ll E_F$ only particles

next to the Fermi surface participate in the response of the system to external perturbations. The energy width of the distribution function $n(\mathbf{p})$ where this happens is $\sim T$. The situation is similar in the Fermi liquid, so that the least important energy scale is T . The quantum uncertainty of the quasiparticle energy is $\sim \hbar/\tau$ where τ is the relaxation time of the quasiparticles. So, we should have the inequality

$$\frac{\hbar}{\tau} \ll T. \quad (3.23)$$

As we understand, only quasiparticles at energies within the width $\sim T$ next to the Fermi surface can scatter, and they stay in the same energy interval. Since the scattering occurs in binary collisions, we have

$$\frac{1}{\tau} \sim T^2, \quad (3.24)$$

and the inequality (3.23) is surely satisfied at $T \rightarrow 0$. For a strongly interacting Fermi liquid at $T \rightarrow 0$ all energy parameters are $\sim \epsilon_F$. Hence, Eq. (3.24) may be written as $\tau^{-1} \sim T^2/\hbar|\epsilon_F|$, and the condition (3.23) is equivalent to

$$T \ll |\epsilon_F|. \quad (3.25)$$

Equations (3.24) and (3.25) are consistent with our discussion of the decay rate τ_E^{-1} of an excitation with energy E in a weakly interacting Fermi gas in the previous subsection. We found Eq. (3.14), and the finite-temperature relaxation rate that we discuss now is obtained by putting $E \sim T$. Then, with $\sigma = 4\pi a^2$, $v_F = p_F/m$, $p_F \sim \hbar n^{1/3}$, and $E \sim T$, equation (3.14) gives $\tau^{-1} \sim (na^3)^{2/3} T^2/\hbar E_F$. Approaching strong interactions one has $na^3 \sim 1$, so that we have $\tau^{-1} \sim T^2/\hbar E_F$ and the condition (3.23) reduces to $T \ll E_F$.

For low-temperature distributions which are close to the step-wise distribution (3.22), to first approximation one can take $n(\mathbf{p})$ in the form (3.22) for calculating $\epsilon(\mathbf{p})$. Then the latter becomes a definite function of p . Next to the Fermi surface where $\epsilon(p)$ makes sense, it can be expanded in powers of $(p - p_F)$. Confining ourselves to the linear term we have

$$\epsilon - \epsilon_F \simeq v_F(p - p_F), \quad (3.26)$$

where

$$\mathbf{v}_F = \left. \frac{\partial \epsilon}{\partial \mathbf{p}} \right|_{p=p_F} \quad (3.27)$$

is the velocity of quasiparticles at the Fermi surface, and we may introduce an effective mass of a quasiparticle:

$$m^* = \frac{p_F}{v_F}. \quad (3.28)$$

Let us now briefly discuss the interaction between quasiparticles. The deviation δn of the distribution function $n(\mathbf{p})$ from the step-wise behavior (3.22)

achieved for non-interacting quasiparticles, leads to a change in the quasiparticle energy. Assuming that both quantities are spin independent we have

$$\delta\epsilon(\mathbf{p}) = \int f(\mathbf{p}, \mathbf{p}') \delta n(\mathbf{p}') \frac{d^3 p'}{(2\pi\hbar)^3}. \quad (3.29)$$

The function $f(\mathbf{p}, \mathbf{p}')$ is called *interaction function* of quasiparticles. Then, near the Fermi surface the quasiparticle energy can be written as

$$\epsilon(\mathbf{p}) - \epsilon_F = v_F(p - p_F) + \int f(\mathbf{p}, \mathbf{p}') \delta n(\mathbf{p}') \frac{d^3 p'}{(2\pi\hbar)^3}. \quad (3.30)$$

The deviation δn is significantly different from zero only near the Fermi surface. For this reason we may write

$$f(\mathbf{p}, \mathbf{p}') = \frac{\pi^2 \hbar^3}{2m^* p_F} F(\vartheta), \quad (3.31)$$

where ϑ is the angle between \mathbf{p} and \mathbf{p}' . Taking into account the exchange interaction one adds the term $\boldsymbol{\sigma} \cdot \boldsymbol{\sigma}' G(\vartheta)$ to $F(\vartheta)$, where $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}'$ are Pauli matrices acting on spin variables corresponding to the momentum variables \mathbf{p} and \mathbf{p}' . Note that in the two-component liquid f represents the second variational derivative of the total energy with respect to the distribution functions of the components.

The interaction function f is a very important quantity and the knowledge of this function allows one to find many observables. In particular, using f one can establish a relation between the true particle mass m and the effective mass m^* . This is done in *Problem 1* to this *Lecture*. Here we consider a simple example of the weakly interacting two-component Fermi gas. To first order in the perturbation theory the total energy is

$$E_0 = \sum_{\mathbf{p}} \frac{p^2}{2m} (n_{\uparrow}(\mathbf{p}) + n_{\downarrow}(\mathbf{p})) + \frac{g}{V} \sum_{\mathbf{p}, \mathbf{p}'} n_{\uparrow}(\mathbf{p}) n_{\downarrow}(\mathbf{p}').$$

Then, for the quasiparticle energies we have:

$$\begin{aligned} \epsilon_{\uparrow}(p) &= \frac{\delta E_0}{\delta n_{\uparrow}} = \frac{p^2}{2m} + \frac{g}{V} \sum_{\mathbf{p}} n_{\downarrow}(\mathbf{p}) = \frac{p^2}{2m} + n_{\downarrow} g = \mu_{\uparrow} + \frac{(p^2 - p_F^2)}{2m}, \\ \epsilon_{\downarrow}(p) &= \frac{\delta E_0}{\delta n_{\downarrow}} = \frac{p^2}{2m} + n_{\uparrow} g = \mu_{\downarrow} + \frac{(p^2 - p_F^2)}{2m}. \end{aligned}$$

The interaction function is given by

$$f = \frac{V \delta^2 E_0}{\delta n_{\uparrow} \delta n_{\downarrow}} = g.$$

3.3 Hydrodynamic regime

Non-equilibrium states of the Fermi liquid are described by distribution functions which depend not only on momenta, but also on coordinates and time. Such functions $n(\mathbf{p}, \mathbf{r}, t)$ obey the kinetic equation:

$$\frac{dn}{dt} = Stn, \quad (3.32)$$

where Stn is the so-called collisional integral determining the rate of change of the quasiparticle number in a given element of the phase volume, due to their collisions with each other. For only slightly non-equilibrium states we have:

$$n(\mathbf{p}, \mathbf{r}, t) = n_0(\mathbf{p}) + \delta n(\mathbf{p}, \mathbf{r}, t), \quad (3.33)$$

and the quasiparticle energy is $\epsilon = \epsilon_0 + \delta\epsilon$, where

$$\frac{\partial\epsilon}{\partial\mathbf{r}} = \frac{\partial\delta\epsilon}{\partial\mathbf{r}} = \int f(\mathbf{p}, \mathbf{p}') \frac{\partial\delta n(\mathbf{p}', \mathbf{r}, t)}{\partial\mathbf{r}} \frac{d^3p'}{(2\pi\hbar)^3}. \quad (3.34)$$

The full time derivative of $n(\mathbf{p}, \mathbf{r}, t)$ is expressed through partial derivatives as

$$\frac{dn}{dt} = \frac{\partial\delta n}{\partial t} + \frac{\partial\delta n}{\partial\mathbf{r}} \dot{\mathbf{r}} + \frac{\partial n}{\partial\mathbf{p}} \dot{\mathbf{p}}. \quad (3.35)$$

For quasiparticles the role of the Hamiltonian function is played by ϵ . So, the Hamiltonian equations read:

$$\dot{\mathbf{r}} = \frac{\partial\epsilon}{\partial\mathbf{p}}; \quad (3.36)$$

$$\dot{\mathbf{p}} = -\frac{\partial\epsilon}{\partial\mathbf{r}}. \quad (3.37)$$

Then, confining ourselves to zero and linear order terms in δn we obtain:

$$\frac{\partial\delta n}{\partial t} + \frac{\partial\epsilon_0}{\partial\mathbf{p}} \frac{\partial\delta n}{\partial\mathbf{r}} - \frac{\partial\delta\epsilon}{\partial\mathbf{r}} \frac{\partial n_0}{\partial\mathbf{p}} = Stn. \quad (3.38)$$

Equations (3.36), (3.37), and (3.38) are classical. Hence, the de Broglie wavelength of quasiparticles \hbar/p_F should be much smaller than the characteristic distance L at which the distribution function changes: $L \gg \hbar/p_F$. So, the frequency of change of the distribution function, which is $\omega \sim v_F/L$, should satisfy the condition:

$$\hbar\omega \ll |\epsilon_F|. \quad (3.39)$$

Let us now discuss the limiting case where

$$\omega\tau \ll 1, \quad (3.40)$$

with $\tau^{-1} \propto T^2$ being the relaxation rate of quasiparticles. The criterion (3.40) corresponds to the *hydrodynamic regime*. Equation (3.40) is equivalent to the

condition $l \ll \lambda$, where l is the mean free path of quasiparticles, and λ is their wavelength. In this case quasiparticle collisions establish a local thermal equilibrium in every elementary volume of the system. This means that we are dealing with ordinary *hydrodynamic waves* propagating with velocity

$$u = \sqrt{\frac{\partial P}{\partial \rho}}, \quad (3.41)$$

where P is the pressure, ρ is the mass density, and at a finite temperature the partial derivative should be taken at a constant entropy. These waves have the dispersion relation

$$\omega = uk, \quad (3.42)$$

with k being their wavevector.

For the weakly interacting Fermi gas, where τ^{-1} is given by Eq. (3.14), the inequality (3.40) requires extremely small frequencies:

$$\omega \ll n\sigma v_F \left(\frac{T}{E_F}\right)^2 \sim (na^3)^{2/3} \left(\frac{T}{E_F}\right) T.$$

For realistic densities in the range from 10^{12} to 10^{14} cm^{-3} and typical values of the scattering length $a \sim 100$ \AA , assuming $T \sim 0.1E_F$ we obtain that ω should be smaller than T by more than 4 orders of magnitude. At commonly used temperatures ranging from 100 nK to 1 μK this requires frequencies $\omega \ll 10$ s^{-1} . So, it is rather difficult to reach the hydrodynamic regime in the weakly interacting Fermi gas.

3.4 Collisionless regime. Zero sound

If the condition opposite to Eq. (3.40) is satisfied, i.e.

$$\omega\tau \gg 1, \quad (3.43)$$

then there is no equilibrium in each small volume of the system, and quasiparticle collisions do not play a role. This regime is identified as *collisionless*, and there is another type of waves called *zero sound*.

So, under the condition (3.43) we may neglect the collisional integral in the kinetic equation (3.38) and write it in the form:

$$\frac{\partial \delta n}{\partial t} + \mathbf{v} \frac{\partial \delta n}{\partial \mathbf{r}} - \frac{\partial \delta \epsilon}{\partial \mathbf{r}} \frac{\partial n_0}{\partial \mathbf{p}} = 0, \quad (3.44)$$

where $\mathbf{v} = \partial \epsilon_0 / \partial \mathbf{p}$ is the unperturbed quasiparticle velocity, so that $\mathbf{v} = v_F \mathbf{n}$ with \mathbf{n} being a unit vector. Taking $n_0 = \theta(p_F - p)$ from Eq. (3.22) we have

$$\frac{\partial n_0}{\partial \mathbf{p}} = -\mathbf{n} \delta(p - p_F) = -\mathbf{v} \delta(\epsilon - \epsilon_F). \quad (3.45)$$

Then, assuming that

$$\delta n = \delta(\epsilon - \epsilon_F) \nu(\mathbf{n}) \exp(i\mathbf{k}\mathbf{r} - i\omega t), \quad (3.46)$$

where $\nu(\mathbf{n})$ is an unknown function, and taking

$$\frac{\partial \delta \epsilon}{\partial \mathbf{r}} = \int f(\mathbf{p}, \mathbf{p}') \frac{\partial \delta n(\mathbf{p}', \mathbf{r}, t)}{\partial \mathbf{r}} \frac{d^3 p'}{(2\pi\hbar)^3}, \quad (3.47)$$

we reduce Eq. (3.44) to

$$(\omega - v_F \mathbf{n}\mathbf{k}) \nu(\mathbf{n}) = \frac{(\mathbf{n}\mathbf{k}) p_F^2}{(2\pi\hbar)^3} \int f(\mathbf{n}, \mathbf{n}') \nu(\mathbf{n}') dO_{\mathbf{p}'}, \quad (3.48)$$

where \mathbf{n} and \mathbf{n}' are unit vectors in the directions of \mathbf{p} and \mathbf{p}' .

We now select \mathbf{k} as a polar axis, and let θ, ϕ be the polar and azimuthal angles of \mathbf{n} . Introducing the velocity of zero sound $u_0 = \omega/k$, from Eq. (3.48) we have:

$$\left(\frac{u_0}{v_F} - \cos \theta \right) \nu(\theta, \phi) = \cos \theta \int F(\varphi) \nu(\theta', \phi') \frac{dO_{\mathbf{p}'}}{4\pi}, \quad (3.49)$$

where φ is the angle between \mathbf{p} and \mathbf{p}' , and the function $F(\varphi)$ is given by Eq. (3.31). Rewriting Eq. (3.49) as

$$\tilde{\nu}(\theta, \phi) = \cos \theta \int \frac{F(\varphi) \tilde{\nu}(\theta', \phi')}{u_0/v_F - \cos \theta'} \frac{dO_{\mathbf{p}'}}{4\pi}, \quad (3.50)$$

where $\tilde{\nu} = (u_0/k - \cos \theta) \nu$, we see that

$$u_0 > v_F, \quad (3.51)$$

otherwise there is a pole in Eq. (3.50) and ω will have an imaginary part for real k .

The zero sound waves deform the Fermi surface. Assuming that $f = f_0 = \text{const}$ ($F = F_0 = \text{const}$) we make sure that the Fermi surface becomes a surface of rotation elongated in the direction of the wave propagation. In this case we obtain

$$\nu = \text{const} \frac{\cos \theta}{u_0/k - \cos \theta}. \quad (3.52)$$

Substituting Eq. (3.52) into Eq. (3.49) we find

$$F_0 \int_0^\pi \frac{\cos \theta}{u_0/k - \cos \theta} \frac{2\pi \sin \theta d\theta}{4\pi} = 1. \quad (3.53)$$

This gives

$$\frac{1}{F_0} = \frac{u_0}{2v_F} \ln \left(\frac{u_0 + v_F}{u_0 - v_F} \right). \quad (3.54)$$

For the weakly (repulsively) interacting two-component Fermi gas we have $f = g$ and $m^* = m$ (see *Problem 1* to this *Lecture*). This yields

$$F = \frac{p_F m}{2\pi^2 \hbar^3} g = \frac{2}{\pi} \frac{p_F a}{\hbar} \ll 1. \quad (3.55)$$

Then, from Eq. (3.54) we obtain:

$$u_0 \approx v_F \left[1 + 2 \exp\left(-\frac{\pi \hbar}{p_F a}\right) \right]. \quad (3.56)$$

Thus, the velocity of zero sound in the weakly interacting Fermi gas almost coincides with the Fermi velocity.

Problems 3

3.1 Establish a relation between the true particle mass m and the effective mass m^* using the interaction function of quasiparticles.

The velocity of a quasiparticle is $\partial\epsilon/\partial\mathbf{p}$. Hence, the flux of quasiparticles is $\int n(\partial\epsilon/\partial\mathbf{p})d^3p/(2\pi\hbar)^3$. Since the number of quasiparticles coincides with the number of particles, the transfer of mass by quasiparticles is a product of their flux and the particle mass m . So, we have

$$\int \mathbf{p} n(\mathbf{p}) \frac{d^3p}{(2\pi\hbar)^3} = \int m \frac{\partial\epsilon}{\partial\mathbf{p}} n(\mathbf{p}) \frac{d^3p}{(2\pi\hbar)^3}. \quad (3.57)$$

Making a variation we obtain:

$$\begin{aligned} \int \mathbf{p} \delta n(\mathbf{p}) \frac{d^3p}{(2\pi\hbar)^3} &= m \int \frac{\partial\epsilon}{\partial\mathbf{p}} \delta n(\mathbf{p}) \frac{d^3p}{(2\pi\hbar)^3} + m \int n(\mathbf{p}) \frac{\delta\partial\epsilon(\mathbf{p})}{\partial\mathbf{p}} \frac{d^3p}{(2\pi\hbar)^3} \\ &= m \int \frac{\partial\epsilon}{\partial\mathbf{p}} \delta n(\mathbf{p}) \frac{d^3p}{(2\pi\hbar)^3} - m \int f(\mathbf{p}, \mathbf{p}') \frac{\partial n(\mathbf{p}')}{\partial\mathbf{p}'} \delta n(\mathbf{p}) \frac{d^3p d^3p'}{(2\pi\hbar)^6}, \end{aligned}$$

where we integrated by parts in the first term of the first line and used Eq.(3.29). As the quantity δn is arbitrary we find:

$$\frac{\mathbf{p}}{m} = \frac{\partial\epsilon}{\partial\mathbf{p}} - \int f(\mathbf{p}, \mathbf{p}') \frac{\partial n(\mathbf{p}')}{\partial\mathbf{p}'} \frac{d^3p'}{(2\pi\hbar)^3}.$$

Taking a step-wise distribution $n(\mathbf{p}) = \theta(p_F - p)$, using Eq. (3.26) for the quasiparticle energy, and putting $\mathbf{p} = p_F \mathbf{n}$ where \mathbf{n} is a unit vector, we then have

$$\frac{p_f \mathbf{n}}{m} = v_F \mathbf{n} + \int \delta(p_F - p) \mathbf{n}' dO_{\mathbf{n}'} \frac{p^2 dp}{(2\pi\hbar)^3}.$$

Multiplying both sides of this equation by \mathbf{n} , integrating over the solid angle $dO_{\mathbf{n}}$, and dividing both sides by p_F , we obtain:

$$\frac{1}{m} = \frac{1}{m^*} + \frac{p_F}{(2\pi)^2 \hbar^3} \int_0^\pi f(\varphi) \cos \varphi \sin \varphi d\varphi, \quad (3.58)$$

where φ is the angle between \mathbf{p} and \mathbf{p}' .

For the weakly interacting two-component Fermi gas we have $f = g = \text{const}$, and Eq. (3.58) gives $m^* = m$.

3.2 Consider a two-component Fermi gas with repulsive interaction between particles in an external harmonic potential $V(r) = m\omega^2 r^2/2$ at $T = 0$. Find the density distribution.

Lecture 4. Attractively interacting Fermi gas. Superfluid pairing

4.1 Cooper problem

This *Lecture* is dedicated to a weakly interacting two-component Fermi gas with attractive interaction between fermions of different components. We will discuss the phenomenon of superfluid pairing and superfluid phase transition.

In order to get a physical intuition we first consider two attractively interacting particles in vacuum. The wavefunction of their relative motion obeys the Schroedinger equation

$$\left\{ -\frac{\hbar^2}{m} \Delta_{\mathbf{r}} + U(\mathbf{r}) \right\} \psi(\mathbf{r}) = E\psi(\mathbf{r}), \quad (4.1)$$

where \mathbf{r} is the relative coordinate. Representing the wavefunction $\psi(\mathbf{r})$ as

$$\psi(\mathbf{r}) = \sum_{\mathbf{k}} c_{\mathbf{k}} \exp(i\mathbf{k}\mathbf{r}), \quad (4.2)$$

we have $\sum_{\mathbf{k}'} c_{\mathbf{k}'} [E - 2E_{k'} - U(\mathbf{r})] \exp(i\mathbf{k}'\mathbf{r}) = 0$ with $E_k = \hbar^2 k^2/2m$, and obtain a set of equations for the coefficients $c_{\mathbf{k}}$:

$$(E - 2E_k) c_{\mathbf{k}} = \frac{1}{\Omega} \sum_{\mathbf{k}'} U(\mathbf{k} - \mathbf{k}') c_{\mathbf{k}'}, \quad (4.3)$$

where

$$U(\mathbf{k} - \mathbf{k}') = \int d\mathbf{r} U(\mathbf{r}) \exp[i(\mathbf{k}' - \mathbf{k})\mathbf{r}], \quad (4.4)$$

and Ω is the volume.

We now use the model where

$$U(\mathbf{k} - \mathbf{k}') = \begin{cases} -V_0; & E_{k'} \leq \tilde{\omega} \\ 0, & \text{otherwise} \end{cases} \quad (4.5)$$

Then we obtain

$$c_{\mathbf{k}} = \frac{-V_0}{E - 2E_k} \frac{1}{\Omega} \sum_{\mathbf{k}'} c_{\mathbf{k}'} \theta(\tilde{\omega} - E_{k'}),$$

where the θ -function is equal to unity for positive values of the argument and zero for negative arguments. Multiplying both sides of this equation by $\theta(\tilde{\omega} - E_k)$ and making a summation over \mathbf{k} we arrive at the relation;

$$\sum_{\mathbf{k}} c_{\mathbf{k}} \theta(\tilde{\omega} - E_k) = \frac{1}{\Omega} \sum_{\mathbf{k}} \frac{-V_0 \theta(\tilde{\omega} - E_k)}{E - 2E_k} \sum_{\mathbf{k}'} c_{\mathbf{k}'} \theta(\tilde{\omega} - E_{k'})$$

or

$$-\frac{1}{\Omega} \sum_{\mathbf{k}} \frac{V_0 \theta(\tilde{\omega} - E_k)}{E - 2E_k} = 1. \quad (4.6)$$

Searching for the solution with a negative energy, that is $E = -2\Delta < 0$, we find:

$$\frac{V_0}{\Omega} \sum_{\mathbf{k}} \frac{\theta(\tilde{\omega} - E_k)}{2\Delta + 2E_k} = \int_0^{k_0} \frac{V_0}{(\hbar^2 k^2/m + 2\Delta)} \frac{4\pi k^2 dk}{(2\pi)^3} = 1,$$

where $k_0 = \sqrt{2m\tilde{\omega}}/\hbar$. This gives

$$\frac{mV_0 k_0}{2\pi^2 \hbar^2} \left\{ 1 - \frac{\sqrt{2m\Delta}}{\hbar k_0} \arctan\left(\frac{\hbar k_0}{\sqrt{2m\Delta}}\right) \right\} = 1. \quad (4.7)$$

Assuming that $\Delta \ll \tilde{\omega}$ and, hence, $k_0 = \sqrt{2m\tilde{\omega}}/\hbar \gg \sqrt{2\Delta m}/\hbar$, we have

$$\lambda \left(1 - \frac{\pi}{2} \sqrt{\frac{\Delta}{\tilde{\omega}}} \right) = 1, \quad (4.8)$$

where

$$\lambda = \frac{mk_0}{2\pi^2 \hbar^2} V_0. \quad (4.9)$$

Equation (4.8) clearly shows that a bound state of two particles may exist only for $\lambda > 1$, which is a known result.

It is now easy to turn to the famous Cooper problem which lies in the basis of superfluid pairing phenomenon. Consider two attractively interacting fermions on top of a filled Fermi sphere, so that the states with wavevectors $k < k_F$ are filled and thus blocked for the occupation. Assume that the center of mass of the two fermions is at rest and, hence, their wavefunction is

$$\psi(\mathbf{r}) = \sum_{|\mathbf{k}| > k_F} c_{\mathbf{k}} \exp(i\mathbf{k}\mathbf{r}), \quad (4.10)$$

with \mathbf{r} being the separation between these fermions. We thus return to the already discussed problem, where now the coefficients $c_{\mathbf{k}}$ are given by ($|\mathbf{k}| > k_F$):

$$c_{\mathbf{k}} = \frac{1}{E - 2E_k} \sum_{|\mathbf{k}'| > k_F} U(\mathbf{k} - \mathbf{k}') c_{\mathbf{k}'}. \quad (4.11)$$

We then use the following model:

$$U(\mathbf{k} - \mathbf{k}') = \begin{cases} -V_0; & E_k > E_F, E_F < E_{k'} \leq E_F + \tilde{\omega} \\ 0, & \text{otherwise} \end{cases} \quad (4.12)$$

We search for the solution in the form $E = 2E_F - 2\Delta$ ($\Delta > 0$), i.e. for the bound state of the two fermions on top of the filled Fermi sphere and assume that the inequality $\Delta \ll \tilde{\omega} \ll E_F$ is satisfied. Then we have:

$$\int_{k_F}^{k'_0} \frac{V_0}{(2\Delta + 2E_k - 2E_F)} \frac{4\pi k^2 dk}{(2\pi)^3} = 1, \quad (4.13)$$

where $k'_0 = \sqrt{2m(E_F + \tilde{\omega})/\hbar^2} \approx k_F + m\tilde{\omega}/\hbar^2 k_F$, and $(m\tilde{\omega}/\hbar^2 k_F) \ll k_F$. Equation (4.13) then yields:

$$\frac{mV_0 k_F^2}{2\pi^2 \hbar^2} \int_0^{m\tilde{\omega}/\hbar^2 k_F} \frac{d(k - k_F)}{(2m\Delta/\hbar^2) + 2k_F(k - k_F)} = \frac{\lambda}{2} \ln\left(\frac{\tilde{\omega}}{\Delta}\right) = 1. \quad (4.14)$$

So, we thus obtain

$$\Delta = \tilde{\omega} \exp\left\{-\frac{2}{\lambda}\right\} \neq 0. \quad (4.15)$$

In other words, two attractively interacting fermions on top of the filled Fermi sphere always form sort of bound pairs. This phenomenon is called *superfluid pairing*, and we will see later in the course how it leads to superfluidity. Note that it is a collective phenomenon. The presence of the filled Fermi sphere is crucial.

4.2 BCS approach. Gapped single-particle excitations

We now consider a weakly (attractively) interacting gas of \uparrow and \downarrow fermions with densities $n_\uparrow = n_\downarrow = n$. The grand-canonical Hamiltonian of this system is obtained by subtracting the term $\mu\hat{N}$ (\hat{N} is the operator of the total number of particles) from \hat{H} (3.1) and it reads:

$$\hat{H} = \sum_{\mathbf{k}; \sigma=\uparrow, \downarrow} \xi_k \hat{a}_{\sigma\mathbf{k}}^\dagger \hat{a}_{\sigma\mathbf{k}} + \frac{g}{V} \sum_{\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3} \hat{a}_{\uparrow\mathbf{k}_3}^\dagger \hat{a}_{\downarrow\mathbf{k}_4}^\dagger \hat{a}_{\downarrow\mathbf{k}_2} \hat{a}_{\uparrow\mathbf{k}_1}, \quad (4.16)$$

where $g = 4\pi\hbar^2 a/m < 0$, the chemical potential is $\mu = \hbar^2 k_F^2/2m$, and we introduced the notation

$$\xi_k = \frac{\hbar^2 k^2}{2m} - \mu = \frac{\hbar^2}{2m}(k^2 - k_F^2). \quad (4.17)$$

We will denote the first term of Eq. (4.16) as $\hat{H}_{0\mu}$, and the second (interaction) term as \hat{H}_{int} . In terms of the field operators $\hat{\psi}_\uparrow, \hat{\psi}_\downarrow$, the term $\hat{H}_{0\mu}$ is obtained by subtracting $\mu \sum_\sigma \int \hat{\psi}_\sigma^\dagger(\mathbf{r}) \hat{\psi}_\sigma(\mathbf{r}) d^3r$ from \hat{H}_K (2.30) and reads:

$$\hat{H}_{0\mu} = \int d^3r \sum_\sigma \hat{\psi}_\sigma^\dagger(\mathbf{r}) \left(-\frac{\hbar^2}{2m} \nabla^2 - \mu \right) \hat{\psi}_\sigma(\mathbf{r}). \quad (4.18)$$

The interaction part of the Hamiltonian is given by equation (2.33) and we introduce a significant simplification writing \hat{H}_{int} in the form:

$$\hat{H}_{int} = \int d^3r \{ U(\mathbf{r}) [\hat{\psi}_{\uparrow}^{\dagger}(\mathbf{r})\hat{\psi}_{\uparrow}(\mathbf{r}) + \hat{\psi}_{\downarrow}^{\dagger}(\mathbf{r})\hat{\psi}_{\downarrow}(\mathbf{r})] + \Delta(\mathbf{r})\hat{\psi}_{\uparrow}^{\dagger}(\mathbf{r})\hat{\psi}_{\downarrow}^{\dagger}(\mathbf{r}) + \Delta^*(\mathbf{r})\hat{\psi}_{\downarrow}(\mathbf{r})\hat{\psi}_{\uparrow}(\mathbf{r}) \}, \quad (4.19)$$

where the quantities $U(\mathbf{r})$ and $\Delta(\mathbf{r})$ are called the Hartree-Fock potential and pairing potential, respectively. They are given by the relations:

$$U(\mathbf{r}) = g \langle \hat{\psi}_{\uparrow}^{\dagger}(\mathbf{r})\hat{\psi}_{\uparrow}(\mathbf{r}) \rangle = g \langle \hat{\psi}_{\downarrow}^{\dagger}(\mathbf{r})\hat{\psi}_{\downarrow}(\mathbf{r}) \rangle, \quad (4.20)$$

$$\Delta(\mathbf{r}) = g \langle \hat{\psi}_{\downarrow}(\mathbf{r})\hat{\psi}_{\uparrow}(\mathbf{r}) \rangle. \quad (4.21)$$

In free space U and Δ are independent of \mathbf{r} and are real, so that $\Delta = \Delta^*$.

Why do we write Eq. (4.19)? Note that there is Wick's theorem in an ideal gas (however, Δ in this case is zero):

$$\langle \hat{\psi}_{\uparrow}^{\dagger}\hat{\psi}_{\downarrow}^{\dagger}\hat{\psi}_{\downarrow}\hat{\psi}_{\uparrow} \rangle = \langle \hat{\psi}_{\uparrow}^{\dagger}\hat{\psi}_{\uparrow} \rangle \langle \hat{\psi}_{\downarrow}^{\dagger}\hat{\psi}_{\downarrow} \rangle + \langle \hat{\psi}_{\uparrow}^{\dagger}\hat{\psi}_{\downarrow}^{\dagger} \rangle \langle \hat{\psi}_{\downarrow}\hat{\psi}_{\uparrow} \rangle + \text{other pairs of averages.}$$

Assuming weak interactions we simply kept in mind this theorem and used a partial average in Eq. (4.19).

It is also important that the Hartree-Fock potential can be absorbed in the chemical potential by making a transformation $\mu \rightarrow \mu - U$, and it essentially drops out of the problem. We thus may restrict ourselves to the interaction Hamiltonian

$$\hat{H}_{int} = \Delta \int d^3r [\hat{\psi}_{\uparrow}^{\dagger}(\mathbf{r})\hat{\psi}_{\downarrow}^{\dagger}(\mathbf{r}) + \hat{\psi}_{\downarrow}(\mathbf{r})\hat{\psi}_{\uparrow}(\mathbf{r})]. \quad (4.22)$$

Then the momentum-space grand-canonical Hamiltonian (4.16) takes the form:

$$\hat{H}_{BCS} = \sum_{\mathbf{k}; \sigma=\uparrow,\downarrow} \xi_{\mathbf{k}} \hat{a}_{\sigma\mathbf{k}}^{\dagger} \hat{a}_{\sigma\mathbf{k}} + \Delta \sum_{\mathbf{k}} (\hat{a}_{\downarrow\mathbf{k}} \hat{a}_{\uparrow-\mathbf{k}} + \hat{a}_{\uparrow-\mathbf{k}}^{\dagger} \hat{a}_{\downarrow\mathbf{k}}^{\dagger}). \quad (4.23)$$

The approach based on the Hamiltonian (4.23) belongs to Bardeen, Cooper, and Schrieffer and is commonly identified as BCS approach.

The Hamiltonian (4.23) is bilinear in the particle operators $\hat{a}_{\sigma,\mathbf{k}}$ and hence can be reduced to a diagonal form by using a canonical Bogoliubov transformation. We employ the Bogoliubov transformation in the form:

$$\hat{b}_{\uparrow\mathbf{k}} = u_{\mathbf{k}} \hat{a}_{\uparrow\mathbf{k}} + v_{\mathbf{k}} \hat{a}_{\downarrow-\mathbf{k}}^{\dagger}, \quad (4.24)$$

$$\hat{b}_{\downarrow\mathbf{k}} = u_{\mathbf{k}} \hat{a}_{\downarrow\mathbf{k}} - v_{\mathbf{k}} \hat{a}_{\uparrow-\mathbf{k}}^{\dagger}, \quad (4.25)$$

where the new (quasiparticle) operators $\hat{b}_{\sigma\mathbf{k}}$ satisfy the same (anti)commutation relations as the particle operators $\hat{a}_{\sigma\mathbf{k}}$:

$$\hat{b}_{\sigma\mathbf{k}} \hat{b}_{\sigma'\mathbf{k}'}^{\dagger} + \hat{b}_{\sigma'\mathbf{k}'}^{\dagger} \hat{b}_{\sigma\mathbf{k}} = \delta_{\sigma\sigma'} \delta_{\mathbf{k}\mathbf{k}'}, \quad (4.26)$$

$$\hat{b}_{\sigma\mathbf{k}} \hat{b}_{\sigma'\mathbf{k}'} + \hat{b}_{\sigma'\mathbf{k}'} \hat{b}_{\sigma\mathbf{k}} = 0. \quad (4.27)$$

From Eq. (4.24) we then have

$$\{\hat{b}_{\sigma\mathbf{k}}\hat{b}_{\sigma'\mathbf{k}'}^\dagger\} = u_k u_{k'} \{\hat{a}_{\sigma\mathbf{k}}\hat{a}_{\sigma'\mathbf{k}'}^\dagger\} + (2\delta_{\sigma\sigma'} - 1)v_k v_{k'} \{\hat{a}_{\sigma'\mathbf{-k}}^\dagger \hat{a}_{\sigma\mathbf{-k}'}\} = (u_k^2 + v_k^2)\delta_{\mathbf{k}\mathbf{k}'}\delta_{\sigma\sigma'},$$

where the symbol $\{ \}$ stands for the anticommutation: $\{\hat{a}_{\sigma\mathbf{k}}\hat{a}_{\sigma'\mathbf{k}'}^\dagger\} = \hat{a}_{\sigma\mathbf{k}}\hat{a}_{\sigma'\mathbf{k}'}^\dagger + \hat{a}_{\sigma'\mathbf{k}'}^\dagger\hat{a}_{\sigma\mathbf{k}}$. We thus obtain the normalization condition

$$u_k^2 + v_k^2 = 1. \quad (4.28)$$

The inverse Bogoliubov transformation reads:

$$\hat{a}_{\uparrow\mathbf{k}} = u_k \hat{b}_{\uparrow\mathbf{k}} - v_k \hat{b}_{\downarrow\mathbf{-k}}^\dagger, \quad (4.29)$$

$$\hat{a}_{\downarrow\mathbf{k}} = u_k \hat{b}_{\downarrow\mathbf{k}} + v_k \hat{b}_{\uparrow\mathbf{-k}}^\dagger. \quad (4.30)$$

Then, for reducing \hat{H}_{BCS} (4.23) to the diagonal form

$$\hat{H}_{\text{BCS}} = E_0 + \sum_{\sigma,\mathbf{k}} \epsilon_k \hat{b}_{\sigma\mathbf{k}}^\dagger \hat{b}_{\sigma\mathbf{k}}, \quad (4.31)$$

the functions u_k, v_k should satisfy the Bogoliubov-de Gennes equations

$$\xi_k u_k + \Delta v_k = \epsilon_k u_k, \quad (4.32)$$

$$-\xi_k v_k + \Delta u_k = \epsilon_k v_k. \quad (4.33)$$

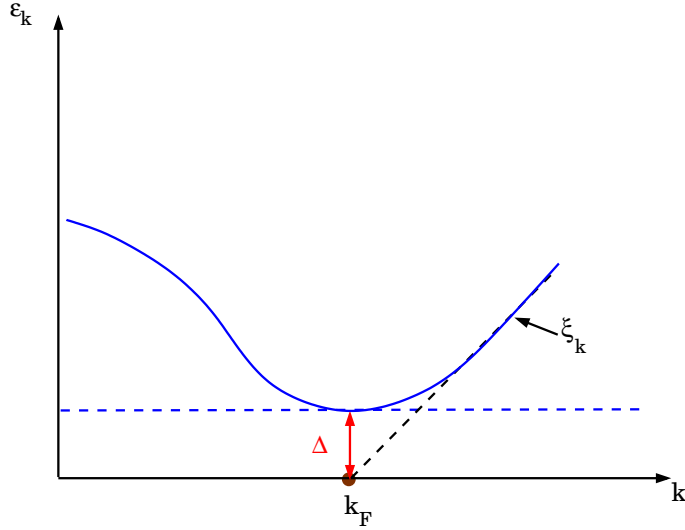


Figure 4.1: Spectrum of single-particle excitations for an attractively interacting two-component Fermi gas.

Taking into account the normalization condition (4.28), equations (4.32) and (4.33) yield:

$$u_k^2 = \frac{1}{2} \left(1 + \frac{\xi_k}{\epsilon_k} \right), \quad (4.34)$$

$$v_k^2 = \frac{1}{2} \left(1 - \frac{\xi_k}{\epsilon_k} \right), \quad (4.35)$$

and lead to the dispersion relation

$$\epsilon_k = \sqrt{\xi_k^2 + \Delta^2}. \quad (4.36)$$

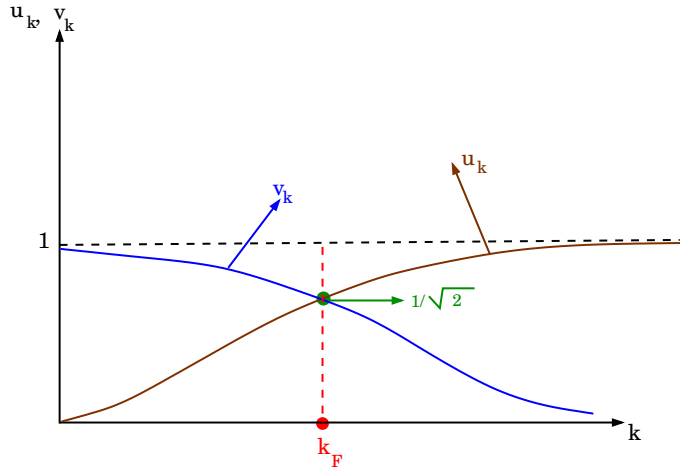


Figure 4.2: The functions u_k (brown curve) and v_k (blue curve).

These quantities are shown in Fig. 4.1 and Fig. 4.2. Thus, the spectrum of single-particle excitations has a gap. This implies that the density of states for the excitations has a singularity at $\epsilon = \Delta$ as shown in Fig. 4.3:

$$\nu(\epsilon) = \int \delta(\epsilon - \sqrt{\xi_k^2 + \Delta^2}) \frac{d^3k}{(2\pi)^3} = \nu(E_F) \frac{\epsilon}{\sqrt{\epsilon^2 - \Delta^2}}; \quad |\epsilon| > \Delta, \quad (4.37)$$

where $\nu(E_F) = mk_F/2\pi^2\hbar^2$ is the particle density of states at the Fermi surface.

4.3 Order parameter and transition temperature

We now have to find the gap Δ . It is given by equation (4.21) and is, therefore, also termed *order parameter*. We write Eq. (4.21) in the form:

$$\begin{aligned} \Delta &= g \langle \hat{\psi}_\downarrow(\mathbf{r}) \hat{\psi}_\uparrow(\mathbf{r}) \rangle = \frac{g}{V} \sum_{\mathbf{k}} \langle \hat{a}_{\downarrow-\mathbf{k}} \hat{a}_{\uparrow\mathbf{k}} \rangle = \frac{g}{V} \sum_{\mathbf{k}} \langle (u_k \hat{b}_{\downarrow-\mathbf{k}} + v_k \hat{b}_{\uparrow\mathbf{k}}^\dagger) (u_k \hat{b}_{\uparrow\mathbf{k}} - v_k \hat{b}_{\downarrow-\mathbf{k}}^\dagger) \rangle \\ &= \frac{g}{V} \sum_{\mathbf{k}} (u_k v_k \langle (\hat{b}_{\uparrow\mathbf{k}}^\dagger \hat{b}_{\uparrow\mathbf{k}} - \hat{b}_{\downarrow-\mathbf{k}} \hat{b}_{\downarrow-\mathbf{k}}^\dagger) \rangle) = -\frac{g}{V} \sum_{\mathbf{k}} u_k v_k (1 - 2N_k). \end{aligned} \quad (4.38)$$

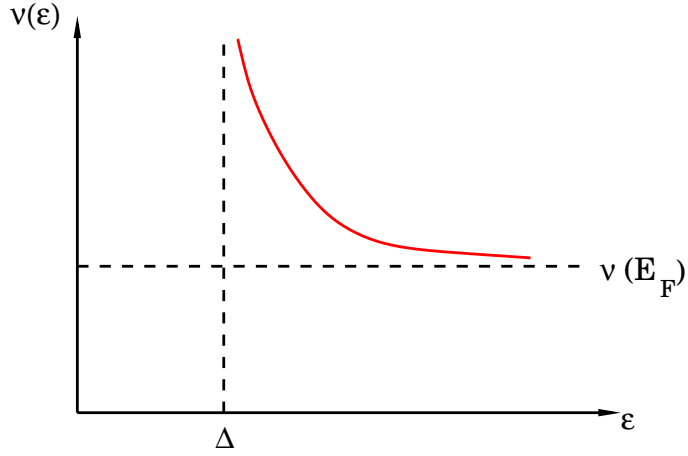


Figure 4.3: Density of states for single-particle excitations.

Single-particle excitations obey Fermi statistics and their occupation numbers N_k are given by the Fermi-Dirac relation:

$$N_k = \langle \hat{b}_{\sigma\mathbf{k}}^\dagger \hat{b}_{\sigma\mathbf{k}} \rangle = \frac{1}{\exp(\epsilon_k/T) + 1}.$$

Using the relation $u_k v_k = \Delta/2\epsilon_k$ following from Eqs. (4.34) and (4.35) we then transform Eq. (4.38) to

$$|g| \int \frac{\tanh(\epsilon_k/2T)}{2\epsilon_k} \frac{d^3k}{(2\pi)^3} = 1. \quad (4.39)$$

Equation (4.39) is called the *gap equation*. At $T = 0$ it becomes

$$|g| \int \frac{1}{2\sqrt{\Delta_0^2 + \xi_k^2}} \frac{d^3k}{(2\pi)^3} = 1. \quad (4.40)$$

The main contribution to the integral in Eq. (4.40) comes from momenta k close to k_F and we may write $|\xi_k| \simeq \hbar v_F |k - k_F| \ll E_F$. We thus obtain:

$$|g| \nu(E_F) \int_0^{\tilde{\omega}} \frac{d\xi}{\sqrt{\xi^2 + \Delta_0^2}} = \lambda \ln \left(\frac{\tilde{\omega}}{\Delta_0} \right) = 1, \quad (4.41)$$

where recalling that $g = 4\pi\hbar^2 a/m$ we have $\lambda = 2k_F |a|/\pi$, and the high-energy cut-off $\tilde{\omega}$ may be put of the order of E_F . This gives the zero-temperature gap:

$$\Delta_0 \approx E_F \exp(-1/\lambda) = E_F \exp(-\pi/2k_F |a|) \ll E_F. \quad (4.42)$$

Let us now return to the initial gap equation (4.39). The highest temperature at which this equation has a non-trivial solution is the critical temperature T_c .

Only below this temperature the spectrum of single-particle excitations has a gap. The calculation of T_c and $\Delta(T)$ is presented in *Problem 1* to this *Lecture*. The critical temperature is related to the zero-temperature gap as

$$T_c = 0.57\Delta_0, \quad (4.43)$$

and the temperature dependence of the gap is given by

$$\Delta = \Delta_0 \left[1 - \sqrt{\frac{2\pi T}{\Delta_0}} \exp(-\Delta_0/T) \right]; \quad T \ll \Delta_0, \quad (4.44)$$

$$\Delta = 3.06T_c \left(1 - \frac{T}{T_c} \right)^{1/2}; \quad T \rightarrow T_c. \quad (4.45)$$

The dependence $\Delta(T)$ is displayed in Fig. 4.4.

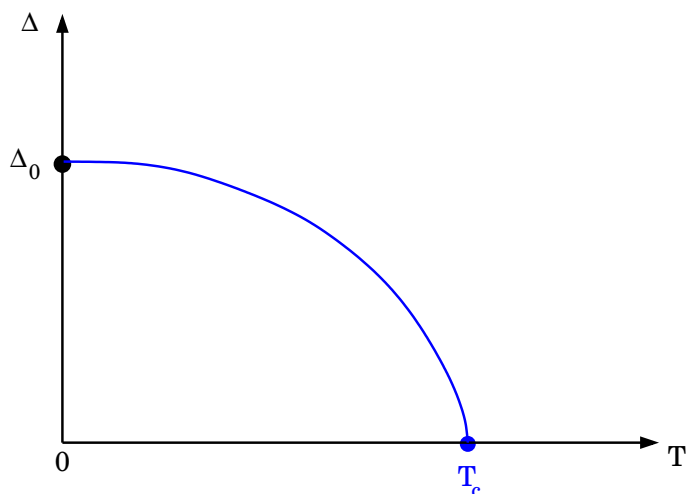


Figure 4.4: The gap in the spectrum of single-particle excitations as a function of temperature.

The spectrum of single-particle excitations, $\epsilon(k)$, satisfies the Landau criterion of superfluidity discussed in *Lecture 6* of *Part 1* of the course. The minimum value of $\epsilon(k)/k$ is non-zero. Therefore, the weakly interacting two-component Fermi gas with intercomponent attraction is superfluid. We will discuss the issue of fermionic superfluidity in *Lecture 5*. Here we only mention that since $\Delta(T)$ decreases with increasing temperature and becomes zero for $T > T_c$, the temperature T_c is the *superfluid transition* temperature. For $T > T_c$ the spectrum is not gapped and is in some sense similar to the spectrum of an ideal gas (Fermi liquid), so that there is no superfluidity.

Note that the picture of pairing near the Fermi surface is better imaged as correlations between states in k -space, which allows two particles to have

zero total momentum. The momentum interval δk in the region of correlations corresponds to the energy interval Δ . So, we have $\delta k \sim \Delta/\hbar v_F$. The related length is $\zeta \sim \hbar v_F/\Delta$. It determines a characteristic distance between particles with correlated momenta. At $T = 0$ it is called *coherence length* and is given by

$$\zeta_0 = \frac{\hbar v_F}{\Delta_0} \sim \frac{1}{k_F} \exp\left(\frac{\pi}{2|k_F|a}\right) \gg n^{-1/3}. \quad (4.46)$$

Problems 4

4.1 Find a relation between the zero-temperature gap Δ_0 and the critical temperature T_c . Calculate $\Delta(T)$ in the limit of ultralow temperatures $T \ll \Delta_0$.

At $T = T_c$ the gap vanishes and the excitation energy is $\epsilon_k = |\xi_k| = \hbar^2|k^2 - k_F^2|/2m$. Thus the gap equation (4.39) becomes:

$$|g| \int \frac{\tanh(|\xi_k|/2T_c)}{2|\xi_k|} \frac{d^3k}{(2\pi)^3} = 1. \quad (4.47)$$

On the other hand, at $T = 0$ we have $\epsilon_k = \sqrt{\xi_k^2 + \Delta_0^2}$ and the gap equation (4.40). Subtracting Eq. (4.40) from Eq. (4.47) and turning to the integration over $d\xi_k$ we obtain:

$$\int_{-E_F}^{\infty} \left[\frac{\tanh(|\xi_k|/2T_c)}{2|\xi_k|} - \frac{1}{2\sqrt{\xi_k^2 + \Delta_0^2}} \right] \frac{mk}{2\pi^2\hbar^2} d\xi_k = 0. \quad (4.48)$$

Assuming that both Δ_0 and T_c are very small we divide the area of integration in Eq. (4.48) into two parts: from $-\omega$ to ω , and from ω to infinity plus from $-E_F$ to $-\omega$. The quantity ω is chosen such that $E_F \gg \omega \gg \Delta_0, T_c$. In the second area, i.e. from ω to ∞ and from $-E_F$ to $-\omega$, we may put $\tanh(|\xi_k|/2T_c) = 1$ and $\Delta_0 = 0$. Hence, the integrand vanishes, and we are left with the integration from $-\omega$ to ω for which we put $k = k_F$ outside the square brackets. Taking into account that the integrand becomes an even function of ξ_k we reduce Eq. (4.48) to

$$\int_0^{\omega} \left[\frac{\tanh(\xi_k/2T_c)}{2\xi_k} - \frac{1}{2\sqrt{\xi_k^2 + \Delta_0^2}} \right] d\xi_k = 0. \quad (4.49)$$

Integrating the first term in the square brackets we turn to the integration variable $\xi_k/2T_c$, and integrating the second term to the integration variable ξ_k/Δ_0 . The integration yields:

$$\ln\left(\frac{\omega}{2T_c}\right) \tanh\left(\frac{\omega}{2T_c}\right) - \int_0^{\omega/2T_c} \frac{\ln x}{\cosh^2 x} dx - \ln\left\{\frac{\omega}{\Delta_0} + \sqrt{1 + \frac{\omega^2}{\Delta_0^2}}\right\} = 0$$

Since $\omega \gg T_c, \Delta_0$, we may put $\tanh(\omega/2T_c) = 1$, set the upper limit of integration over dx equal to infinity, and write the last term as $\ln(2\omega/\Delta_0)$. This gives:

$$\ln \frac{\Delta_0}{4T_c} = \int_0^\infty \frac{\ln x}{\cosh^2 x} dx = \ln \pi - 2 \ln 2 - C,$$

where $C = 0.577$ is the Euler constant. We thus obtain equation (4.43):

$$T_c = \frac{\exp C}{\pi} \Delta_0 = 0.57 \Delta_0.$$

For calculating $\Delta(T)$ at $T \ll \Delta_0$ we subtract the gap equation (4.39) from the zero-temperature gap equation (4.40) and thus obtain:

$$\int_0^\infty \left[\frac{1}{2\sqrt{\xi_k^2 + \Delta_0^2}} - \frac{\tanh(\sqrt{\xi_k^2 + \Delta^2}/2T)}{2\sqrt{\xi_k^2 + \Delta^2}} \right] \frac{k^2 dk}{2\pi^2} = 0.$$

Again, turning to the integration over $d\xi_k$ and dividing the area of integration into two parts: from $-\omega$ to ω , and from ω to ∞ plus from $-E_F$ to $-\omega$, we select ω such that $\Delta_0 \ll \omega \ll E_F$ and find:

$$\int_0^\omega \left[\frac{1}{\sqrt{\xi_k^2 + \Delta_0^2}} - \frac{\tanh(\sqrt{\xi_k^2 + \Delta^2}/2T)}{\sqrt{\xi_k^2 + \Delta^2}} \right] d\xi_k = 0. \quad (4.50)$$

Using a relation

$$\tanh x = 1 - \frac{2 \exp(-x)}{\exp x + \exp(-x)}$$

we reduce equation (4.50) to

$$\ln \frac{\Delta}{\Delta_0} + 2 \int_0^\omega \frac{\exp(-\sqrt{\xi_k^2 + \Delta^2}/T)}{\sqrt{\xi_k^2 + \Delta^2}} d\xi_k = 0,$$

where we took into account that $\omega \gg \Delta_0$. Since $T \ll \Delta, \Delta_0$ the main contribution to the remaining integral over $d\xi_k$ comes from $\xi_k \ll \Delta$. We then represent $\sqrt{\xi_k^2 + \Delta^2}$ in the exponent as $\sqrt{\xi_k^2 + \Delta^2} = \Delta + \xi_k^2/2\Delta$ and put $\xi_k = 0$ in the denominator of the integrand. After the integration over $d\xi_k$ we have:

$$\ln \frac{\Delta_0}{\Delta} = \sqrt{\frac{2\pi T}{\Delta}} \exp\left(-\frac{\Delta}{T}\right). \quad (4.51)$$

Writing $\Delta = \Delta_0 - \delta\Delta$ and assuming that $\delta\Delta \ll T$, from Eq. (4.51) we obtain:

$$\delta\Delta = \sqrt{2\pi\Delta_0 T} \exp\left(-\frac{\Delta_0}{T}\right).$$

This equation justifies the assumption that $\delta\Delta \ll T$. So, the final result is given by equation (4.44):

$$\Delta(T) = \Delta_0 \left[1 - \sqrt{\frac{2\pi T}{\Delta_0}} \exp\left(-\frac{\Delta_0}{T}\right) \right].$$

4.2 Calculate $\Delta(T)$ at temperatures near T_c , where $\Delta \ll T$ and $(T_c - T) \ll T_c$.

Lecture 5. Superfluidity in Fermi gases

5.1 Landau criterion in Fermi gases

In this *Lecture* we discuss the phenomenon of superfluidity in attractively interacting Fermi gases. As we have established in *Lecture 4*, the spectrum of single-particle excitations is $\epsilon(p) = \sqrt{\Delta^2 + \xi_p^2}$, where $\xi_p = v_F(p - p_F)$ (p is here the true momentum, not the wavevector). The gap Δ is exponentially small in the limit of weak interactions and becomes zero for $T > T_c = 0.57\Delta_0$, with Δ_0 being the zero-temperature gap. In Fig. 5.1 we compare the spectra of attractively and repulsively interacting Fermi gases. As we see, there is a drastic difference. For the attractively interacting gas there is a minimum value $(\epsilon(p)/p)_{\min} > 0$, whereas for repulsive interactions it is zero.

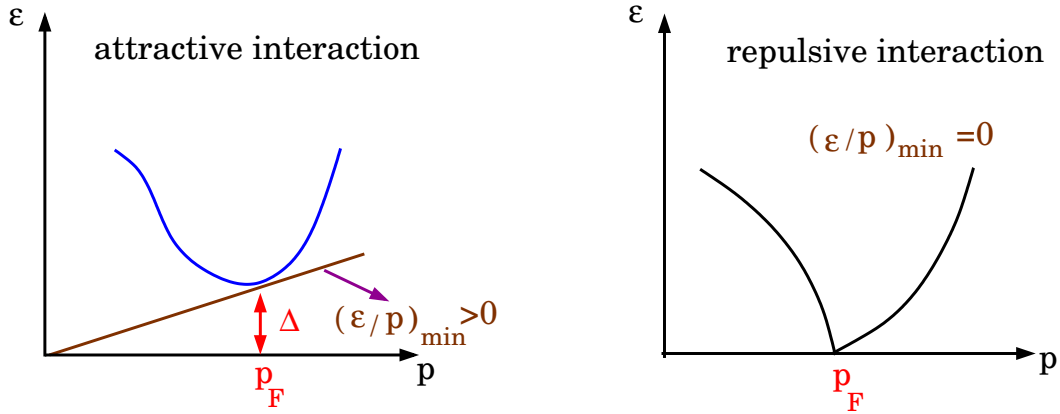


Figure 5.1: Spectrum of single-particle excitations for attractively and repulsively interacting Fermi gases.

Let us repeat the arguments given in *Lecture 6* of *Part 1* of the course, now on support of superfluidity in attractively interacting Fermi gases. Imagine that an excitation with energy $\epsilon(p)$ and momentum \mathbf{p} appears in such a system. Then the energy and momentum of the gas become $\epsilon(p)$ and $\hbar\mathbf{p}$. The excitation appears, for example, if the gas is in a capillary and the latter is moving with velocity $-\mathbf{v}$. Then, in the reference frame where the capillary is at rest but the gas is moving with velocity \mathbf{v} , we have the energy and momentum:

$$E = \epsilon(p) + \mathbf{p}\mathbf{v} + \frac{Mv^2}{2},$$

$$\mathbf{p}' = \mathbf{p} + M\mathbf{v},$$

where M is the mass of the gas. The term $Mv^2/2$ is the initial kinetic energy of the gas, and $\epsilon + \mathbf{p}\mathbf{v}$ is the energy change due to the appearance of the excitation. So, at $T = 0$ we should have $\epsilon + \mathbf{p}\mathbf{v} < 0$, otherwise the excitation can not

appear. This is possible only if $v > (\epsilon/p)$. Thus, if there is a minimum value $(\epsilon/p)_{min} = v_c$, for velocities $v < v_c$ the excitations do not appear. Then, there is no friction between the capillary and the gas, and one has *superfluidity*.

So, the superfluidity requires the condition

$$\left(\frac{\epsilon(p)}{p}\right)_{min} > 0,$$

and the velocity of the motion

$$v < v_c = \left(\frac{\epsilon(p)}{p}\right)_{min}.$$

For the attractively interacting Fermi gas we have a very small critical velocity:

$$v_c \approx \frac{\Delta}{mv_F} \sim v_F \exp\left(-\frac{\pi}{2k_F|a|}\right). \quad (5.1)$$

5.2 Superfluid current

Let us now discuss the superfluid current. First of all, we write the Bogoliubov-de Gennes equations for the coordinate-space wavefunctions of the excitations. In free space they read (see *Problem 1* to this *Lecture*):

$$\hat{H}_0 u_\nu + \Delta v_\nu = \epsilon_\nu u_\nu, \quad (5.2)$$

$$-\hat{H}_0 v_\nu + \Delta^* u_\nu = \epsilon_\nu v_\nu, \quad (5.3)$$

where $\hat{H}_0 = (-\hbar^2/2m)(d^2/d\mathbf{r}^2) - \mu$, and the index ν labels quantum states of the excitations. Assuming that in the presence of current the order parameter (gap) is complex we write

$$\Delta = \bar{\Delta} \exp(2i\phi). \quad (5.4)$$

Comparing equations (5.2) and (5.3) with the Bogoliubov-de Gennes equations for a Bose-condensed gas (see *Lecture 5* of *Part 1*) we see that Δ plays a role of the "wavefunction of a condensate of Cooper pairs". We now write

$$u_\nu = \bar{u}_\nu \exp(i\phi), \quad (5.5)$$

$$v_\nu = \bar{v}_\nu \exp(-i\phi). \quad (5.6)$$

For small gradients of ϕ and vanishingly low momenta of the excitations the quantities \bar{u} , \bar{v} coincide with u_k/\sqrt{V} , v_k/\sqrt{V} where u_k , v_k are given by Eqs. (4.34) and (4.35). This is clear from the Bogoliubov-de Gennes equations obtained after substituting u_ν (5.5) and v_ν (5.6) into Eqs. (5.2) and (5.3):

$$\begin{aligned} \hat{H}_0 \bar{u}_\nu + \bar{\Delta} \bar{v}_\nu + \text{gradients of } \phi &= \epsilon_\nu \bar{u}_\nu, \\ -\hat{H}_0 \bar{v}_\nu + \bar{\Delta} \bar{u}_\nu + \text{gradients of } \phi &= \epsilon_\nu \bar{v}_\nu. \end{aligned}$$

The operator of the current is given by

$$\hat{\mathbf{j}} = -\frac{i\hbar}{2m} \sum_\sigma \left[\hat{\psi}_\sigma^\dagger \nabla \hat{\psi}_\sigma - (\nabla \hat{\psi}_\sigma^\dagger) \hat{\psi}_\sigma \right]. \quad (5.7)$$

Since the field operators $\hat{\psi}_\sigma$ can be expressed as (see *Problem 1* to this *Lecture*):

$$\begin{aligned}\hat{\psi}_\uparrow &= \sum_\nu (u_\nu \hat{b}_{\uparrow\nu} - v_\nu^* \hat{b}_{\downarrow\nu}^\dagger), \\ \hat{\psi}_\downarrow &= \sum_\nu (u_\nu \hat{b}_{\downarrow\nu} + v_\nu^* \hat{b}_{\uparrow\nu}^\dagger),\end{aligned}$$

at $T = 0$ we obtain the following expectation value:

$$\langle j \rangle = -\frac{i\hbar}{2m} \sum_{\sigma\nu} (v_\nu \nabla v_\nu^* - v_\nu^* \nabla v_\nu),$$

which in the uniform case transforms to

$$\langle j \rangle = \frac{\hbar}{m} \left(\sum_{\sigma\mathbf{k}} \frac{v_k^2}{V} \right) \nabla\phi. \quad (5.8)$$

Let us now recall the expression for v_k in the uniform case. It is given by Eq. (4.35). Except for a narrow vicinity of k_F , where the presence of the gap Δ in the expression for the excitation energy is important, we have $v_k = 1$ for $k < k_F$ and $v_k = 0$ for $k > k_F$. Thus, equation (5.8) gives:

$$\langle j \rangle = \frac{\hbar}{m} \nabla\phi \sum_\sigma \int_0^{k_F} \frac{k^2}{2\pi^2} dk = \frac{\hbar}{m} n \nabla\phi. \quad (5.9)$$

where $n = \sum_\sigma n_\sigma$ is the total density. So, the quantity

$$\mathbf{v}_s = \frac{\hbar}{m} \nabla\phi \quad (5.10)$$

is the superfluid velocity.

The superfluid current is

$$\mathbf{j}_s = n \mathbf{v}_s. \quad (5.11)$$

Thus, at $T = 0$ the whole mass of the gas (liquid) is superfluid.

5.3 Bogoliubov-Anderson sound

We now consider hydrodynamic equations for the superfluid Fermi gas. They read:

$$\frac{\partial n}{\partial t} + \text{div}(n\mathbf{v}) = 0, \quad (5.12)$$

$$m \frac{\partial \mathbf{v}}{\partial t} + \nabla \left[\frac{1}{2} m v^2 + \mu(n) \right] = 0. \quad (5.13)$$

We then write $n = \bar{n} + \delta n$, where \bar{n} is the mean value of the density. Assuming small velocities \mathbf{v} and density fluctuations δn , we confine ourselves only to terms

that are linear in \mathbf{v} and δn . Writing $\mathbf{v} = (\hbar/m)\nabla\phi$ we thus obtain linearized hydrodynamic equations:

$$\frac{\partial\delta n}{\partial t} + \nabla \cdot \left(\frac{\hbar\bar{n}}{m} \nabla\phi \right) = 0, \quad (5.14)$$

$$\hbar \frac{\partial\nabla\phi}{\partial t} + \left. \frac{\partial\mu}{\partial n} \right|_{n=\bar{n}} \nabla\delta n = 0. \quad (5.15)$$

Taking the time derivative in Eq. (5.14) and substituting $\partial\nabla\phi/\partial t$ from Eq. (5.15) we obtain:

$$\frac{\partial^2\delta n}{\partial t^2} = \left. \frac{\bar{n}}{m} \frac{\partial\mu}{\partial n} \right|_{n=\bar{n}} \nabla^2\delta n. \quad (5.16)$$

The derivative of the chemical potential with respect to the density is given by

$$\left. \frac{\partial\mu}{\partial n} \right|_{n=\bar{n}} = \left. \frac{\partial}{\partial n} \left(\frac{\hbar^2}{2m} (3\pi^2 n)^{2/3} \right) \right|_{n=\bar{n}} = \frac{2E_F}{3\bar{n}}.$$

Writing small fluctuations of the density as

$$\delta n \propto \exp(-i\epsilon t/\hbar + i\mathbf{k}\mathbf{r})$$

we then find

$$\left(\frac{\epsilon}{\hbar} \right)^2 = \frac{2E_F}{3m} k^2,$$

or

$$\epsilon = \hbar \sqrt{\frac{2E_F}{3m}} k = \frac{\hbar v_F}{\sqrt{3}} k. \quad (5.17)$$

We thus see that there is one more branch of excitations. It is related to fluctuations of the phase of Δ . These excitations obey the Bose statistics and are called Bogoliubov-Anderson sound. Note that for this branch of excitations we have $(\epsilon/\hbar k)_{min} = v_F/\sqrt{3}$ and they do not destroy superfluidity. They do not even change the critical velocity.

5.4 Superfluid and normal density. Thermodynamic quantities near T_c

At a finite temperature the superfluid gas can be implicitly divided into two parts: *superfluid* and *normal*. Repeating the arguments given in *Lecture 6 of Part 1* of the course we now imagine that the "gas of excitations" is moving with respect to the liquid (gas) with velocity \mathbf{v} . The distribution function for the excitations is then obtained by replacing the excitation energy ϵ with $\epsilon - \mathbf{p}\mathbf{v}$, where \mathbf{p} is the excitation momentum. So, the total momentum of the "gas of excitations" per unit volume will be:

$$\mathbf{P} = \int \mathbf{p} N(\epsilon - \mathbf{p}\mathbf{v}) \frac{V d^3 p}{(2\pi\hbar)^3}. \quad (5.18)$$

Assuming a small velocity \mathbf{v} we expand the integrand of Eq. (5.18) in powers of $(\mathbf{p}\mathbf{v})$. Retaining only the linear term we have:

$$\mathbf{P} = - \int \mathbf{p}(\mathbf{p}\mathbf{v}) \frac{dN(\epsilon_p)}{d\epsilon_p} \frac{V d^3p}{(2\pi\hbar)^3}.$$

Averaging over the directions of \mathbf{p} we then obtain:

$$\mathbf{P} = -\frac{\mathbf{v}}{3} \int \frac{dN(\epsilon_p)}{d\epsilon_p} p^2 \frac{V d^3p}{(2\pi\hbar)^3}. \quad (5.19)$$

The gas of excitations can collide with the walls, exchange energy and momentum, and eventually it will be stopped by the walls. The mass density of the liquid (gas) can be written as a sum of the superfluid and normal parts:

$$\rho = \rho_s + \rho_n. \quad (5.20)$$

The expression for the normal density follows from Eq. (5.19) treating \mathbf{P} as the momentum of the system as a whole:

$$\rho_n = -\frac{1}{3} \int \frac{dN(\epsilon_p)}{d\epsilon_p} p^2 \frac{d^3p}{(2\pi\hbar)^3}. \quad (5.21)$$

As we see, there are two contributions to ρ_n : the contribution of gapped single-particle excitations, and at sufficiently low temperatures we should take into account the contribution of Bogoliubov-Anderson sound. Both contributions are zero at $T = 0$. At a finite temperature for the contribution of single-particle gapped excitations we have:

$$\rho_n = -\frac{2}{3} \int p^2 \frac{d}{d\epsilon_p} \left(\frac{1}{\exp(\epsilon_p/T) + 1} \right) \frac{d^3p}{(2\pi\hbar)^3}, \quad (5.22)$$

where $\epsilon = \sqrt{\Delta^2 + [(p^2 - p_F^2)/2m]^2}$, and an additional factor 2 comes from the summation over \uparrow and \downarrow excitation branches. The main contribution to the integral comes from momenta p close to the Fermi momentum p_F , and we obtain:

$$\rho_n = -\frac{p_F^4}{3\pi^2 v_F \hbar^3} \int_{-\infty}^{\infty} \frac{dN(\epsilon)}{d\epsilon} d\xi, \quad (5.23)$$

with $\xi = v_F(p - p_F)$. Since the total density is $\rho = mp_F^3/3\pi^2$, we can write:

$$\rho_n = -2\rho \int_0^{\infty} \frac{dN(\epsilon)}{d\epsilon} d\xi. \quad (5.24)$$

We first notice that at the superfluid transition temperature, $T = T_c$, we have $\Delta = 0$ and $\epsilon = |\xi|$, so that

$$-2 \int_0^{\infty} \frac{dN}{d\epsilon} d\xi = 1.$$

Just below T_c we may expand the integral in Eq. (5.24) in powers of Δ^2/T_c^2 . This gives

$$\frac{\rho_n}{\rho} = 1 - 2 \frac{\Delta^2}{T_c^2} \frac{7\zeta(3)}{8\pi^2} = 1 - 2 \frac{T_c - T}{T_c}; \quad 0 < (T_c - T) \ll T_c. \quad (5.25)$$

For very low temperatures, $T \ll \Delta_0$, we may write

$$\frac{dN}{d\epsilon} = -\frac{1}{T} \frac{\exp(\epsilon/T)}{(\exp(\epsilon/T) + 1)^2} \approx -\frac{1}{T} \exp\left(-\frac{\Delta_0}{T} - \frac{\xi^2}{2\Delta_0 T}\right),$$

and Eq. (5.24) yields

$$\frac{\rho_n}{\rho} = \left(\frac{2\pi\Delta_0}{T}\right)^{1/2} \exp(-\Delta_0/T); \quad T \ll \Delta_0. \quad (5.26)$$

The normal density decreases exponentially with temperature, and we have to take into account the contribution of bosonic Bogoliubov-Anderson sound.

For these excitations we have the dispersion relation (5.17) and the distribution function

$$N(\epsilon) = \frac{1}{\exp(\epsilon/T) - 1}.$$

Then, equation (5.21) is transformed to

$$\rho_n = \frac{1}{6\pi^2 T} \int_0^\infty \frac{\exp(\epsilon/T)}{(\exp(\epsilon/T) - 1)^2} \frac{p^4 dp}{\hbar^3}.$$

The integration with $\epsilon = v_F p / \sqrt{3}$ is straightforward and it yields:

$$\rho_n = \frac{1}{6\pi^2 T \hbar^3} \left(\frac{\sqrt{3}T}{v_F}\right)^5 \int_0^\infty \frac{x^4 \exp x}{(\exp x - 1)^2} dx = \frac{27\sqrt{3}}{4} \rho \left(\frac{T}{E_F}\right)^4 \zeta(4). \quad (5.27)$$

where $\zeta(4)$ is very close to unity. Thus, at $T \rightarrow 0$ the Bogoliubov-Anderson sound gives the leading contribution to the normal density.

We now show that the BCS superfluid transition is of the second order. Indeed, for $T = T_c$ we have $\Delta = 0$ and $\epsilon(p) = |p^2 - p_F^2|/2m$. This is the same as in the Fermi liquid. We should only renormalize the chemical potential in order to take into account the interactions, not more than that. Thus, the energy and other thermodynamic quantities are continuous at $T = T_c$. So, this is *not* the *first order* transition.

Let us focus on the derivatives of thermodynamic functions, for example on the heat capacity. Variation of the energy under variations of the occupation numbers is

$$\delta E = \sum_{\mathbf{p}} \epsilon(p) (\delta N_{\uparrow\mathbf{p}} + \delta N_{\downarrow\mathbf{p}}) = 2 \sum_{\mathbf{p}} \epsilon(p) \delta N_{\mathbf{p}}. \quad (5.28)$$

Dividing Eq. (5.28) by δT we have

$$C = 2 \int \epsilon(p) \frac{\partial N(p)}{\partial T} \frac{V d^3 p}{(2\pi\hbar)^3} = \frac{V m p_F}{\pi^2} \int_{-\infty}^\infty \epsilon \frac{\partial N(\epsilon)}{\partial T} d\xi, \quad (5.29)$$

where again $\xi = v_F(p - p_F)$ and we took into account that the main contribution to the integral comes from momenta near the Fermi surface.

Using the Fermi-Dirac relation for $N(\epsilon)$ we obtain;

$$\epsilon \frac{\partial N(\epsilon)}{\partial T} = \left[\frac{\epsilon^2}{T^2} - \frac{1}{2T} \frac{\partial \Delta^2}{\partial T} \right] \frac{\exp(\epsilon/T)}{(\exp(\epsilon/T) + 1)^2}. \quad (5.30)$$

The result of the integration of the first term in the square brackets on the right hand side of Eq. (5.30) is continuous at $T = T_c$, and it gives the heat capacity in the normal (non-superfluid) phase, C_n . The second term is discontinuous at $T = T_c$. In the normal phase, i.e. for $T > T_c$, it is equal to zero. In the superfluid phase at $T \rightarrow T_c$, using Eq. (4.45) we have:

$$\frac{\partial \Delta^2}{\partial T} = -T_c \frac{8\pi^2}{7\zeta(3)}; \quad T \rightarrow T_c. \quad (5.31)$$

So, the result of the integration of the second term in the square brackets of Eq. (5.30) gives a jump in the heat capacity at $T = T_c$:

$$C_s - C_n = \frac{4Vmp_F}{7\zeta(3)} \int_{-\infty}^{\infty} \frac{\exp(\epsilon/T)}{(\exp(\epsilon/T) + 1)^2} d\xi. \quad (5.32)$$

Putting $\epsilon = |\xi|$ in the integrand of Eq. (5.32) we find:

$$C_s - C_n = \frac{4Vmp_F}{7\zeta(3)} T_c \sim N \exp\left(-\frac{\pi}{2p_F|a|}\right), \quad (5.33)$$

where N is the total number of particles. We thus clearly see that the superfluid BCS transition is the transition of the second order.

Problems 5

5.1 Derive the Bogoliubov-de Gennes equations for the coordinate-space eigenfunctions of single-particle excitations.

In terms of the field operators $\hat{\psi}_\sigma^\dagger(\mathbf{r})$, $\hat{\psi}_\sigma(\mathbf{r})$ the BCS Hamiltonian is the sum of $\hat{H}_{0\mu}$ (4.18) and \hat{H}_{int} (4.19) with $U(\mathbf{r}) = 0$:

$$\hat{H}_{BCS} = \int d^3r \left[\sum_{\sigma} \hat{\psi}_\sigma^\dagger(\mathbf{r}) (\hat{H}_0 + V(\mathbf{r})) \hat{\psi}_\sigma(\mathbf{r}) + \Delta(\mathbf{r}) \hat{\psi}_\uparrow^\dagger(\mathbf{r}) \hat{\psi}_\downarrow^\dagger(\mathbf{r}) + \Delta^*(\mathbf{r}) \hat{\psi}_\downarrow(\mathbf{r}) \hat{\psi}_\uparrow(\mathbf{r}) \right], \quad (5.34)$$

where

$$\hat{H}_0 = -\frac{\hbar^2}{2m} \nabla^2 - \mu, \quad (5.35)$$

and we also included an external potential $V(\mathbf{r})$. We now write $\hat{\psi}_\sigma(\mathbf{r})$ in the form:

$$\hat{\psi}_\uparrow(\mathbf{r}) = \sum_\nu [u_\nu(\mathbf{r})\hat{b}_{\uparrow\nu} - v_\nu^*(\mathbf{r})\hat{b}_{\downarrow\nu}^\dagger], \quad (5.36)$$

$$\hat{\psi}_\downarrow(\mathbf{r}) = \sum_\nu [u_\nu(\mathbf{r})\hat{b}_{\downarrow\nu} + v_\nu^*(\mathbf{r})\hat{b}_{\uparrow\nu}^\dagger], \quad (5.37)$$

which is nothing else than the Bogoliubov transformation in the coordinate space. Substituting the field operators given by Eqs. (5.36) and (5.37) into H_{BCS} (5.34) we find that the latter reduces to the diagonal form

$$\hat{H}_{\text{BCS}} = E_0 + \sum_{\sigma\nu} \epsilon_\nu \hat{b}_{\sigma\nu}^\dagger \hat{b}_{\sigma\nu}$$

if the functions $u_\nu(\mathbf{r})$, $v_\nu(\mathbf{r})$ satisfy the Bogoliubov-de Gennes equations:

$$[\hat{H}_0 + V(\mathbf{r})]u_\nu(\mathbf{r}) + \Delta(\mathbf{r})v_\nu(\mathbf{r}) = \epsilon_\nu u_\nu(\mathbf{r}), \quad (5.38)$$

$$-[\hat{H}_0 + V(\mathbf{r})]v_\nu(\mathbf{r}) + \Delta^*(\mathbf{r})u_\nu(\mathbf{r}) = \epsilon_\nu v_\nu(\mathbf{r}). \quad (5.39)$$

In this respect, the functions $u_\nu(\mathbf{r})$, $v_\nu(\mathbf{r})$ can be called eigenfunctions of elementary excitations. In the absence of external potential $V(\mathbf{r})$ we immediately see that Eqs. (5.38) and (5.39) coincide with equations (5.2) and (5.3).

If the current is also absent, then putting the quantum number ν of an excitation as its wavevector \mathbf{k} we have

$$u_{\mathbf{k}} = \frac{u_k}{\sqrt{V}} \exp(i\mathbf{k}\mathbf{r}), \quad (5.40)$$

$$v_{\mathbf{k}} = \frac{v_k}{\sqrt{V}} \exp(i\mathbf{k}\mathbf{r}), \quad (5.41)$$

where u_k , v_k are given by Eqs. (4.34) and (4.35). Substituting $u_{\mathbf{k}}$, $v_{\mathbf{k}}$ given by Eqs. (5.40) and (5.41) into Eqs. (5.36) and (5.37) and recalling that $\hat{\psi}_\sigma(\mathbf{r}) = \sum_{\mathbf{k}} \hat{a}_{\sigma\mathbf{k}} \exp(i\mathbf{k}\mathbf{r})$, we arrive at the momentum-space Bogoliubov transformations (4.29), (4.30) and (4.24), (4.25).

5.2 Express the ratio ρ_n/ρ through $\Delta^{-1}d\Delta/dT$ in the entire range of temperatures $T < T_c$. Consider only the contribution of single-particle excitations.

Lecture 6. Ginzburg-Landau approach. Vortices in Fermi gases

6.1 Landau-Ginzburg functional

In this *Lecture* we discuss the Ginzburg-Landau description of (two-component) Fermi superfluids and then turn to vortex structures in these systems. In

previous lectures we introduced and found the order parameter (gap) $\Delta = g\langle\hat{\psi}_\downarrow(\mathbf{r})\hat{\psi}_\uparrow(\mathbf{r})\rangle$. In *Lecture 5* we then showed that at $T = 0$ the gradient of the phase of Δ ($\Delta = \bar{\Delta}\exp(2i\phi)$) is related to the superfluid current $\mathbf{j} = n\mathbf{v}_s$, with n being the total density and $\mathbf{v}_s = (\hbar/m)\nabla\phi$ the superfluid velocity. The quantity $\langle\hat{\psi}_\downarrow(\mathbf{r})\hat{\psi}_\uparrow(\mathbf{r})\rangle$ plays a role of the wavefunction of the "condensate of Cooper pairs". In many cases it is more convenient to introduce the order parameter Ψ in such a way that its phase coincides with the phase of Δ , but the amplitude is $|\Psi|^2 = n_s/2$, where n_s is the superfluid density. Then the superfluid current is

$$\mathbf{j}_s = \frac{\hbar}{2mi}(\Psi\nabla\Psi^* - \Psi^*\nabla\Psi) = \frac{\hbar}{m}n_s\nabla\phi = n_s\mathbf{v}_s. \quad (6.1)$$

At $T = 0$ where $n_s = n$, equation (6.1) exactly coincides with Eq. (5.11) which we derived in *Lecture 5*.

As we have shown, at $T = T_c$ the system undergoes the second order phase transition. The physical nature of this transition lies in an anomalous growth of the fluctuations of the order parameter. Let us consider the spatially uniform system and write the free energy per unit volume as an expansion in powers of Ψ :

$$F = F_n + \alpha(T - T_c)|\Psi|^2 + \frac{b}{2}|\Psi|^4, \quad (6.2)$$

where F_n is the free energy in the normal (non-superfluid) phase. In the presence of currents we have to add the term with the gradient of the order parameter. So, we have

$$F = F_n + \frac{\hbar^2}{4m}|\nabla\Psi|^2 + \alpha(T - T_c)|\Psi|^2 + \frac{b}{2}|\Psi|^4. \quad (6.3)$$

Equation (6.3) is usually called the Ginzburg-Landau functional. The coefficient $\alpha > 0$, so that the superfluid phase is at $T < T_c$. At $T = T_c$ the term $\alpha(T - T_c) = 0$, and the difference between the free energies of superfluid and normal phases is $(F_s - F_n) \propto |\Psi|^4$. The equilibrium value of $|\Psi|^2$ at $T < T_c$ is determined by the minimization of F (6.2) and is equal to

$$|\bar{\Psi}|^2 = \frac{\alpha(T_c - T)}{b}. \quad (6.4)$$

In the uniform case this gives:

$$F_s - F_n = -\frac{\alpha^2}{2b}(T_c - T)^2 \quad (6.5)$$

and leads to the temperature independent heat capacity per unit volume:

$$C_s - C_n = -T \left(\frac{\partial^2(F_s - F_n)}{\partial T^2} \right)_V = \frac{\alpha^2 T_c}{b}, \quad (6.6)$$

as it should be according to our derivation in *Lecture 5*.

For a weakly interacting two-component Fermi gas we have:

$$\alpha = \frac{6\pi^2 T_c}{7\zeta(3)\mu}, \quad (6.7)$$

$$b = \frac{\alpha T_c}{n}. \quad (6.8)$$

This follows from the expression for the energy

$$E = \langle \hat{H} \rangle = \left\langle \sum_{\sigma\mathbf{k}} E_k \hat{a}_{\sigma\mathbf{k}}^\dagger \hat{a}_{\sigma\mathbf{k}} + \Delta \sum_{\mathbf{k}} (\hat{a}_{\downarrow\mathbf{k}} \hat{a}_{\uparrow-\mathbf{k}} + \hat{a}_{\uparrow-\mathbf{k}}^\dagger \hat{a}_{\downarrow\mathbf{k}}^\dagger) \right\rangle$$

using the Bogoliubov transformation (4.29), (4.30) and the expression for the entropy:

$$S = - \sum_{\sigma\mathbf{k}} [N_{\sigma\mathbf{k}} \ln N_{\sigma\mathbf{k}} + (1 - N_{\sigma\mathbf{k}}) \ln(1 - N_{\sigma\mathbf{k}})]. \quad (6.9)$$

Then, the free energy $F = E - TS$ will be given by Eq. (6.2) with α, b from Eqs. (6.7) and (6.8).

6.2 Critical fluctuations

Let us now consider fluctuations of the order parameter. Under a small deviation of $|\Psi|$ from the equilibrium value $|\bar{\Psi}|$ the change of F is

$$\delta F = \frac{1}{2} (|\Psi| - |\bar{\Psi}|)^2 \left(\frac{\partial^2 F}{\partial |\Psi|^2} \right). \quad (6.10)$$

For the quantity $\partial^2 F / \partial |\Psi|^2$ we have:

$$\frac{\partial^2 F}{\partial |\Psi|^2} = 2\alpha(T - T_c); \quad T > T_c, \quad |\bar{\Psi}| = 0. \quad (6.11)$$

$$\frac{\partial^2 F}{\partial |\Psi|^2} = 4\alpha(T_c - T); \quad T < T_c, \quad |\bar{\Psi}| = \frac{\alpha(T_c - T)}{b}. \quad (6.12)$$

The probability of the fluctuation is

$$w \sim \exp \left\{ -\frac{V\delta F}{T} \right\} = \exp \left\{ -\frac{V(|\Psi| - |\bar{\Psi}|)^2}{2T} \frac{\partial^2 F}{\partial |\Psi|^2} \right\}. \quad (6.13)$$

We then obtain the mean square fluctuation of $|\Psi|$:

$$\langle (|\Psi| - |\bar{\Psi}|)^2 \rangle = \frac{T_c}{2\alpha|T - T_c|V} \times \begin{cases} 1; & T > T_c \\ 1/2; & T < T_c \end{cases} \quad (6.14)$$

From Eq. (6.3) we can establish the correlation radius r_c of the fluctuations. It follows from the condition that the gradient term is of the order of the $|\Psi|^2$ term:

$$r_c \sim \sqrt{\frac{\hbar^2}{m\alpha|T_c - T|}}. \quad (6.15)$$

In order to still apply the theory that we discussed in previous lectures, we should have small fluctuations over the volume r_c^3 , compared to $|\bar{\Psi}|^2 = \alpha(T_c - T)/b$:

$$\langle (|\Psi| - |\bar{\Psi}|)^2 \rangle \Big|_{V=r_c^3} \ll |\bar{\Psi}|^2.$$

This leads to the inequality:

$$\frac{T_c}{\alpha|T_c - T|r_c^3} \ll \frac{\alpha}{b}|T_c - T|. \quad (6.16)$$

Substituting r_c from Eq. (6.15) we obtain the famous Ginzburg criterion:

$$\alpha|T_c - T| \gg T_c^2 b^2 \left(\frac{m}{\hbar^2}\right)^3. \quad (6.17)$$

For the weakly interacting Fermi gas, with $\alpha \sim T_c/\mu$; $b \sim T_c^2/\mu n$ from Eqs. (6.7), (6.8) and $\mu = E_F \sim \hbar^2 n^{2/3}/m$, we find:

$$|T_c - T| \gg T_c \left(\frac{T_c}{E_F}\right)^4. \quad (6.18)$$

Thus, in a very narrow interval of temperatures near T_c the theory is not applicable. This interval is called the *region of critical fluctuations*. A theory for this region is beyond the scope of the present course.

6.3 Vortex state

We now turn to the discussion of vortices in the two-component superfluid Fermi gas. What is the vortex state? Assume that in free space the complex order parameter Δ (or Ψ) is of the form

$$\Delta(\mathbf{r}) = \Delta(\rho) \exp(i\phi), \quad (6.19)$$

where $\boldsymbol{\rho}$ is the vector in the $\{x, y\}$ plane and ϕ is the azimuthal angle ($x = \rho \cos \phi$, $y = \rho \sin \phi$). The velocity field \mathbf{v} as derived in *Lecture 5*, then is

$$\mathbf{v} = \frac{\hbar}{2m} \nabla \phi. \quad (6.20)$$

An extra factor 2 in the denominator is related to the fact that now we write the phase ϕ instead of 2ϕ before. Now it is even more convenient, since the factor $2m$ shows that the mass of a Cooper pair is twice the mass of a particle. The velocity field has a tangential form ($\boldsymbol{\rho} \mathbf{v} = 0$):

$$\mathbf{v} = -\frac{[\boldsymbol{\rho} \hat{\mathbf{z}}]}{\rho^2} \frac{\hbar}{2m}, \quad (6.21)$$

where $\hat{\mathbf{z}}$ is a unit vector perpendicular to the $\{x, y\}$ plane. So, we have

$$|\mathbf{v}| = \frac{\hbar}{2m\rho} \quad (6.22)$$

and clearly see that the velocity increases on approach to the vortex line, i.e. to the line where $|\mathbf{v}| \rightarrow \infty$. The circulation is quantized:

$$\oint \mathbf{v} d\mathbf{l} = \frac{\pi \hbar}{m}. \quad (6.23)$$

The angular momentum carried by the vortex is

$$\langle \hat{L}_z \rangle = m \int [\mathbf{r} \times \mathbf{v}] n(\mathbf{r}) d^3 r = \frac{\hbar N}{2}, \quad (6.24)$$

with N being the total number of particles, and $n(\mathbf{r})$ the density distribution. The energy E_v acquired by the vortex is mainly determined by the hydrodynamic kinetic energy $(m/2) \int n v^2 d^3 r$. In a non-rotating cylinder of radius R we have:

$$E_v = \frac{N \hbar}{4mR^2} \ln \left(\frac{R}{\xi} \right), \quad (6.25)$$

where ξ is the radius of the vortex core, which is of the order of the coherence length $\hbar v_F / \Delta$.

Let us now discuss the form of $\Delta(\mathbf{r})$ in the case of the vortex state. For this purpose we write the Bogoliubov-de Gennes equations where we still keep the Hartree-Fock terms:

$$\begin{bmatrix} \mathcal{H} - \mu & \Delta(\mathbf{r}) \\ \Delta^*(\mathbf{r}) & -(\mathcal{H} - \mu) \end{bmatrix} \begin{bmatrix} u_\sigma(\mathbf{r}) \\ v_\sigma(\mathbf{r}) \end{bmatrix} = \epsilon_\sigma \begin{bmatrix} u_\sigma(\mathbf{r}) \\ v_\sigma(\mathbf{r}) \end{bmatrix}, \quad (6.26)$$

where

$$\mathcal{H} = -\frac{\hbar^2}{2m} \nabla^2 - g n(\boldsymbol{\rho}), \quad (6.27)$$

and $g = -4\pi \hbar^2 a / m > 0$ for $a < 0$.

For simplicity we consider $T = 0$ and assume that the system is a very large cylinder and is not rotating. The density profile is given by

$$n(\boldsymbol{\rho}) = \sum_{\sigma \mathbf{k}} |v_{\sigma \mathbf{k}}(\boldsymbol{\rho})|^2 = 2 \sum_{\mathbf{k}} |v_{\mathbf{k}}(\boldsymbol{\rho})|^2, \quad (6.28)$$

and the gap equation reads:

$$\Delta(\boldsymbol{\rho}) = g \sum_{\sigma \mathbf{k}} u_{\sigma \mathbf{k}}(\boldsymbol{\rho}) v_{\sigma \mathbf{k}}^*(\boldsymbol{\rho}). \quad (6.29)$$

The lowest energy solution for which $\Delta \propto \exp(i\phi)$, corresponds to

$$u = u(\rho); \quad v = \tilde{v}(\rho) \exp(-i\phi). \quad (6.30)$$

Since the order parameter is exponentially small in the BCS regime, the density is practically not affected by the vortex. However, Δ vanishes at the vortex line. The solution of this problem is quite cumbersome and requires numerics. Therefore, we only present the dependence $|\Delta(\rho)|$ in Fig. 6.1.

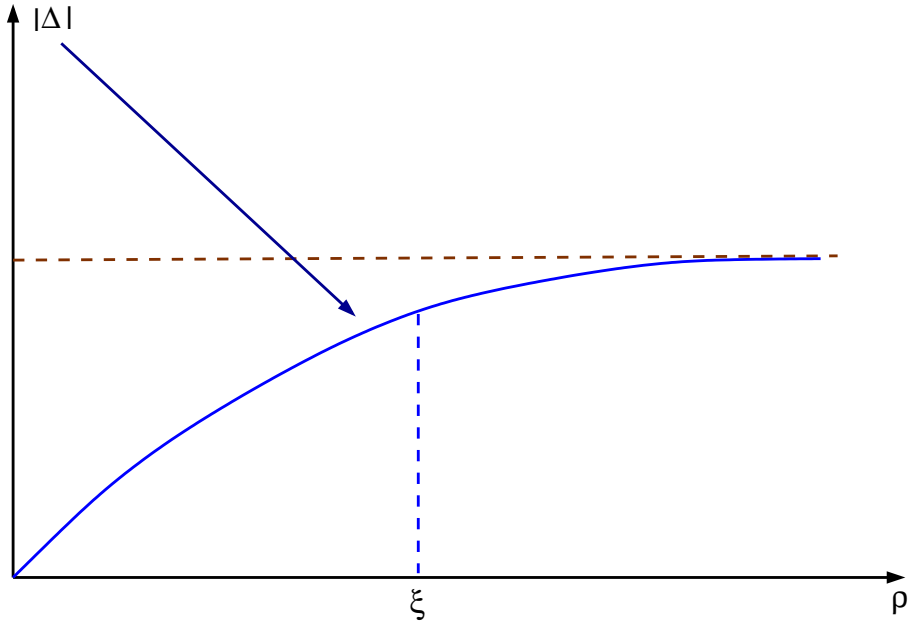


Figure 6.1: Modulus of the order parameter $|\Delta|$ as a function of the distance from the vortex line, ρ .

In a non-rotating cylinder the vortex state is an excited state of the system. We now briefly discuss the case of a rotating cylinder. Let Ω be the rotation frequency. In the rotating reference frame the energy associated with the appearance of the vortex is

$$E = E_v - \Omega \langle L_z \rangle, \quad (6.31)$$

where E_v is given by Eq. (6.25), and $\langle L_z \rangle$ by Eq. (6.24). The critical frequency Ω_c at which the vortex state becomes the ground state of the system is obtained from the condition $E = 0$. We thus have:

$$\Omega_c = \frac{\hbar}{2mR^2} \ln \left(\frac{R}{\xi} \right). \quad (6.32)$$

6.4 Vortices near T_c

Let us now discuss vortices at temperatures close to T_c and employ the Ginzburg-Landau functional (6.3). In the spatially non-uniform system we have to write it in the form:

$$F = \int d^3r \left\{ \frac{\hbar^2}{4m} (\nabla\Psi^* \nabla\Psi) - \alpha(T_c - T)|\Psi|^2 + \frac{b}{2}|\Psi|^4 \right\}. \quad (6.33)$$

The first term in curly brackets can also be written as $-(\hbar^2/4m)\Psi^*\nabla^2\Psi$, which is clearly seen after performing the integration by parts. Let us make a variation

of F . Note that we should vary Ψ^* and Ψ separately. Considering variations $\delta\Psi^*$ we have:

$$\delta F = \int d^3r \delta\Psi^* \left\{ -\frac{\hbar^2}{4m} \nabla^2 \Psi - \alpha(T_c - T)\Psi + b|\Psi|^2 \Psi \right\}.$$

So, setting $\delta F = 0$ we obtain an equation:

$$-\frac{\hbar^2}{4m} \nabla^2 \Psi - \alpha(T_c - T)\Psi + b|\Psi|^2 \Psi = 0. \quad (6.34)$$

Equation (6.34) looks exactly the same as the Gross-Pitaevskii equation for the condensate wavefunction in the case of bosons (see *Lecture 3 of Part 1* of the course). In order to make the analogy explicit we should treat $\alpha(T_c - T)$ as the "chemical potential", and b as the "coupling constant". Since $\alpha \sim T_c/E_F$, $b \sim T_c^2/nE_F$ (see Eqs. (6.7) and (6.8)), we obtain the "healing length":

$$\xi \sim \sqrt{\frac{\hbar^2}{m\mu}} \sim \sqrt{\frac{\hbar E_F}{mT_c(T_c - T)}} \sim \frac{\hbar v_F}{\sqrt{T_c(T_c - T)}} \sim \frac{\hbar v_F}{\Delta},$$

where we took into account that $\Delta(T) \sim \sqrt{T_c(T_c - T)}$ at $T \rightarrow T_c$.

For the vortex state of bosons, Ψ has been obtained in *Lecture 7 of Part 1*. It is presented in Fig. 6.2.

All other solutions at temperatures near T_c are obtained in the same way as in the case Bose-Einstein condensates in *Part 1* of the course.

Problems 6

6.1 Obtain the Ginzburg-Landau functional for the free energy $F = E - TS$ from the microscopic theory for the weakly interacting two-component Fermi gas. Use the following expressions for the energy and entropy:

$$E = 2 \int \epsilon(k) N_{\mathbf{k}} \frac{V d^3k}{(2\pi)^3},$$

$$S = -2 \int [N_{\mathbf{k}} \ln N_{\mathbf{k}} + (1 - N_{\mathbf{k}}) \ln(1 - N_{\mathbf{k}})] \frac{V d^3k}{(2\pi)^3}.$$

Lecture 7. Strongly interacting regime in Fermi gases

7.1 Anomalously large scattering length

We now arrive at the discussion of the so-called strongly interacting regime in dilute Fermi gases. In previous lectures we were writing the interaction term in the Hamiltonian as

$$\hat{H}_{int} = g \int \hat{\psi}_{\uparrow}^{\dagger}(\mathbf{r}) \hat{\psi}_{\downarrow}^{\dagger}(\mathbf{r}) \hat{\psi}_{\downarrow}(\mathbf{r}) \hat{\psi}_{\uparrow}(\mathbf{r}) d^3r,$$

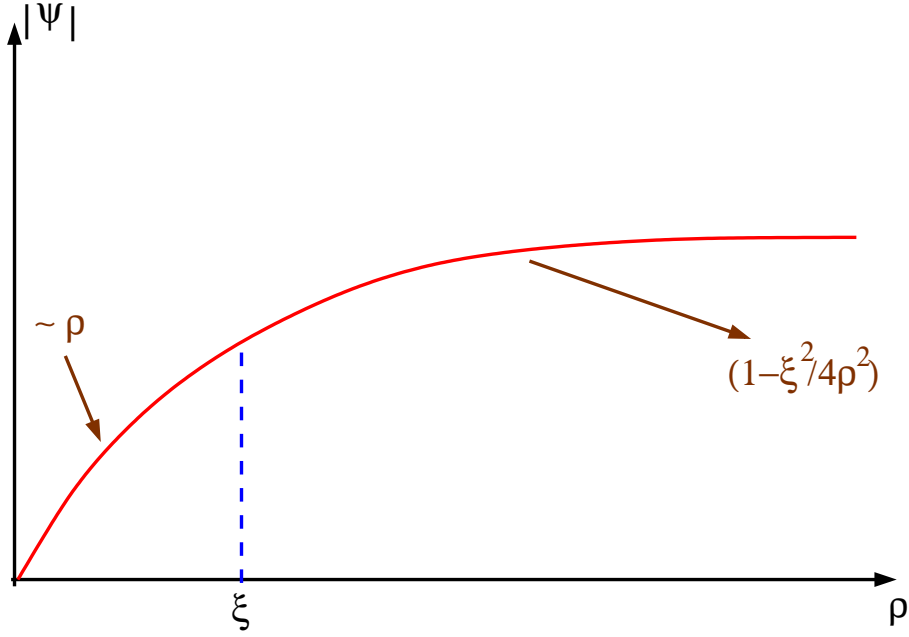


Figure 6.2: Modulus of the order parameter $|\Psi|$ (or $|\Delta|$) as a function of the distance from the vortex core, ρ , at temperatures $T \rightarrow T_C$.

where $g = 4\pi\hbar^2 a/m$, with a being the scattering length for the interaction between \uparrow and \downarrow atoms. We were considering the weakly interacting regime, where $n|a|^3 \ll 1$ and the interaction energy per particle is much smaller than the Fermi energy, so that the interactions can be taken into account within the many-body perturbation theory. What happens if $|a| \rightarrow \infty$, but the gas is still dilute and $nR_e^3 \ll 1$, where R_e is the radius of the interaction potential $U(r)$?

First of all, why and how this can happen? Consider $U(r)$ such that there is a very weakly bound state with zero orbital angular momentum (s -state) and binding energy $\epsilon_0 \rightarrow 0$ (see Fig. 7.1). At $r \gg R_e$ the wavefunction of the bound s -state is governed by the Schroedinger equation for the free motion:

$$-\frac{\hbar^2}{m} \left(\frac{d^2\psi_0}{dr^2} + \frac{2}{r} \frac{d\psi_0}{dr} \right) = -\epsilon_0\psi_0 \quad (7.1)$$

and reads:

$$\psi_0 = C \frac{\exp(-\kappa r)}{r}, \quad (7.2)$$

where $\kappa = \sqrt{m\epsilon_0}/\hbar$, and C is the normalization coefficient.

On the other hand, the wavefunction of continuum states with zero orbital angular momentum and energy $\epsilon \rightarrow 0$ is

$$\psi = \left(1 - \frac{a}{r} \right), \quad (7.3)$$

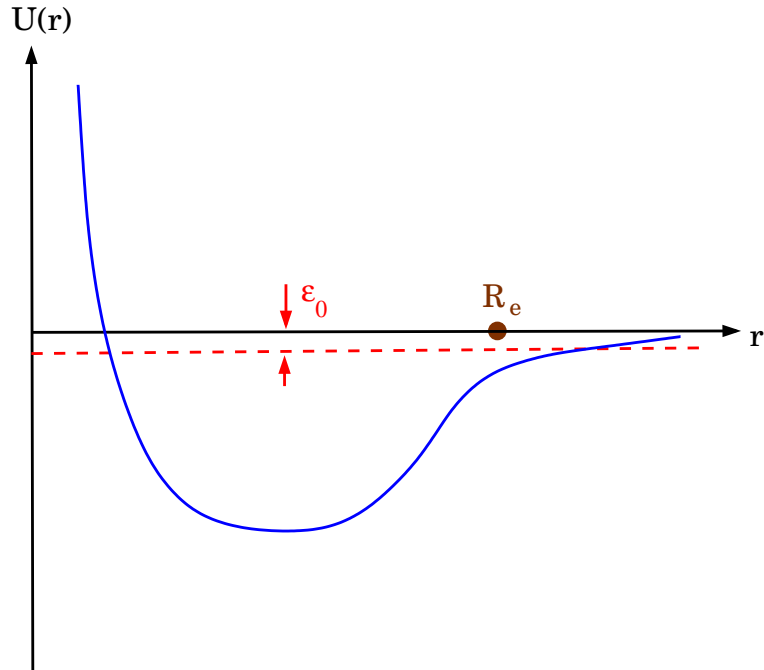


Figure 7.1: Interaction potential $U(r)$ supporting a weakly bound state with zero orbital angular momentum.

where a is the scattering length. In the interval of distances where $\kappa^{-1} \gg r \gg R_e$, equation (7.2) becomes $\psi_0 \propto (1 - 1/\kappa r)$ and it should coincide with Eq. (7.3), except for the normalization coefficient. This is because at such distances in both cases the wavefunction is governed by the Schrodinger equation for free motion with zero energy: $\psi''_{rr} + (2/r)\psi'_r = 0$. So, we thus obtain:

$$a = \frac{1}{\kappa} = \frac{\hbar}{\sqrt{m\epsilon_0}} > 0. \quad (7.4)$$

The scattering length is positive and large ($a \gg R_e$).

Note that we obtained Eq. (7.4) in the presence of a very weakly bound state. The case of a large but negative scattering length corresponds to the presence of the so-called virtual bound state (just make the potential $U(r)$ slightly deeper and the bound state appears). So, we see that one can have $n|a|^3$ large, but $nR_e^3 \ll 1$. For $\epsilon_0 \rightarrow 0$ we have $|a| \rightarrow \infty$.

7.2 Fano-Feshbach resonance

Is there a possibility to manipulate the interaction and modify the scattering length a , thus changing the coupling constant g and influencing the many-body physics? Imagine that there is a collision between two atoms in certain hyperfine

states along the potential curve $U_1(r)$. The hyperfine interaction can change the internal state of one of the colliding atoms or even the internal states of both of them, thus transferring the system to the motion along the potential curve $U_2(r)$ (see Fig. 7.2). Assume now that there is a bound state in the potential $U_2(r)$, which almost coincides with the bottom of continuum states of $U_1(r)$ so that the corresponding energy difference E_0 is small.

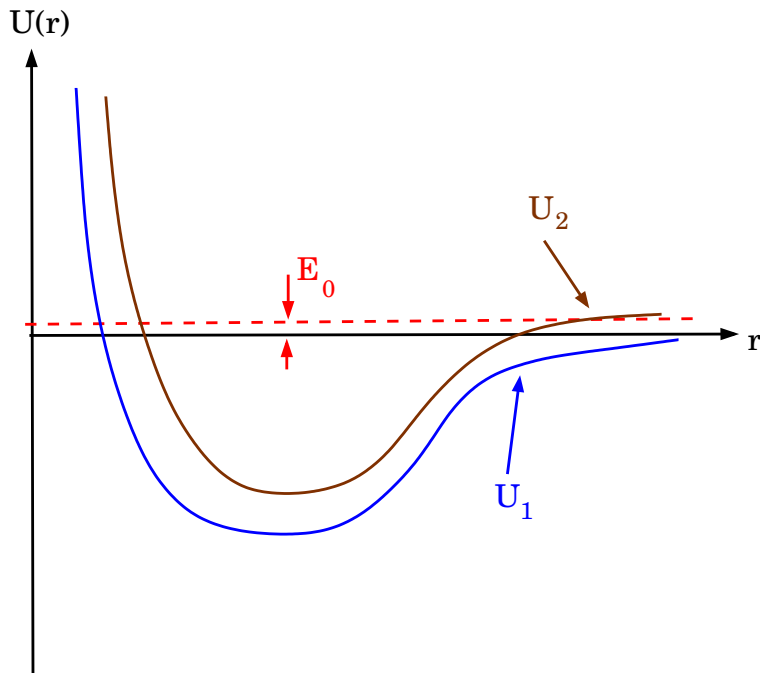


Figure 7.2: The picture of interaction potentials and bound state(s) showing the presence of a Fano-Feshbach resonance.

We then consider the s -wave scattering and write the wavefunction of the scattering state at large distances as

$$\psi(r) = \frac{1}{r} [A \exp(ikr) + B \exp(-ikr)], \quad (7.5)$$

with $k = \sqrt{mE/\hbar^2}$. Treat now the scattering energy E as a complex quantity. This is because the energy of the bound state E_0 is in fact complex and equal to $E_0 - i\Gamma/2$. This state is actually quasibound as it can decay into free atoms, and the quantity Γ is the corresponding width. The condition determining complex eigenstates is that there is no incoming wave at large distances. This means that we should have zero coefficient B for $E = E_0 - i\Gamma/2$:

$$B(E_0 - i\Gamma/2) = 0. \quad (7.6)$$

For the scattering energy E close to E_0 we may expand B in powers of $(E - E_0 + i\Gamma/2)$ and confine ourselves to the linear term of the expansion, which gives

$$B = \left(E - E_0 + i\frac{\Gamma}{2} \right) b,$$

where b is a constant. So, taking into account that $A = B^*$ the wavefunction (7.5) becomes

$$\psi(r) = \frac{1}{r} \left[\left(E - E_0 - i\frac{\Gamma}{2} \right) b^* \exp(ikr) + \left(E - E_0 + i\frac{\Gamma}{2} \right) b \exp(-ikr) \right]. \quad (7.7)$$

The phase δ of this function is given by

$$\exp(2i\delta) = \frac{E - E_0 - i\Gamma/2}{E - E_0 + i\Gamma/2} \exp(2i\delta^{(0)}), \quad (7.8)$$

where $\exp(2i\delta^{(0)}) = -b^*/b$. For $|E - E_0| \gg \Gamma$ we have $\delta = \delta^{(0)}$. So, $\delta^{(0)}$ is the phase far from resonance. For simplicity we put $\delta^{(0)} = 0$, thus omitting the potential scattering. Then, using the well-known formula for the scattering amplitude (see *Lecture 1*):

$$f = \frac{\tan \delta}{k(1 - i \tan \delta)}$$

we obtain:

$$f = -\frac{\Gamma/2}{k(E - E_0 + i\Gamma/2)}. \quad (7.9)$$

For E and E_0 tending to zero the function $B(E)$ can be expanded in powers of E . The point $E = 0$ is the branching point of $B(E)$ so that going from the upper part of the complex half-plane to the lower part transforms B to B^* . Therefore, the expansion of B should be in powers of \sqrt{E} , and we have

$$B(E) = (E - E_0 + i\gamma\sqrt{E})b. \quad (7.10)$$

Thus, we have a relation

$$\frac{\Gamma}{2} = \gamma\sqrt{E} = \gamma\frac{\hbar}{\sqrt{m}}k.$$

We then rewrite equation (7.9) in the form

$$f = -\frac{\hbar\gamma}{\sqrt{2m}(E - E_0 + i\gamma\sqrt{E})},$$

or

$$f = -\frac{1}{a^{-1} + R^*k^2 + ik}, \quad (7.11)$$

where $a^{-1} = -E_0\sqrt{2m}/\hbar\gamma$, and $R^* = \hbar/(\gamma\sqrt{2m})$. For $k \rightarrow 0$ we have $f = -a$ and we see that the scattering length is $a \propto E_0^{-1}$. In our derivation we assumed

that $E_0 > 0$ and obtained $a < 0$. For $E_0 < 0$ the derivation is similar and one obtains $a > 0$. The original description of the discussed resonance scattering belongs to Fano and to Feshbach and it is usually identified as Fano-Feshbach resonance.

The spacing between the potentials $U_1(r)$ and $U_2(r)$ is equal to the difference in the energies of the involved internal (hyperfine) states of the atoms and depends on the magnetic field. As a result, the scattering length becomes field dependent and tends to infinity when $E_0 \rightarrow 0$ (see Fig. 7.3).

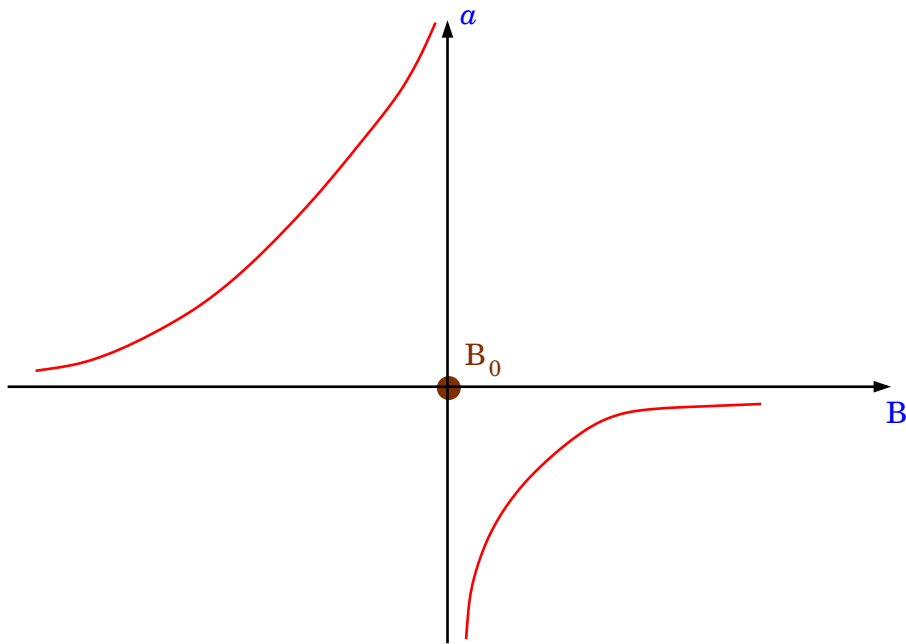


Figure 7.3: The magnetic field dependence of the scattering length for a Fano-Feshbach resonance. The field B_0 corresponds to $|a| \rightarrow \infty$.

7.3 BCS-BEC crossover

Let us now consider $T = 0$ and make a qualitative sketch of the phase diagram. Namely, we find out what kind of ground state we have depending on the value of the scattering length a (or magnetic field B). For a small negative a satisfying the condition $k_F|a| \ll 1$ there is a BCS superfluid that we discussed in the previous lectures. For $|a| \rightarrow \infty$ we have the so-called strongly interacting regime, which we can not treat on the basis of the many-body perturbation theory. For a large positive a still satisfying the inequality $na^3 \ll 1$, \uparrow and \downarrow fermions form a weakly bound state as we discussed in subsection 7.1. So, we have diatomic molecules representing composite bosons, and they should Bose-condense. The wavefunction of each composite boson is given by equation (7.2)

and the corresponding momentum distribution is

$$G(k) = \int \psi_0(\mathbf{r}_1 - \mathbf{r}_2) \exp[i\mathbf{k}(\mathbf{r}_1 - \mathbf{r}_2)] d(\mathbf{r}_1 - \mathbf{r}_2) \propto \frac{1}{a^{-2} + k^2}. \quad (7.12)$$

Considering a Bose-Einstein condensate of the molecules at $T = 0$ we may say that the system consists of real-space pairs of \uparrow and \downarrow fermions, with momentum distribution (7.12). This is an argument on support of the statement that going from BCS (small negative a) to BEC of molecules (positive a and $na^3 \ll 1$) simply transforms Cooper pairs to real space diatomic molecules. The symmetry of the system does not change and we have a *crossover*, not a quantum transition. The phase diagram is displayed in Fig. 7.4

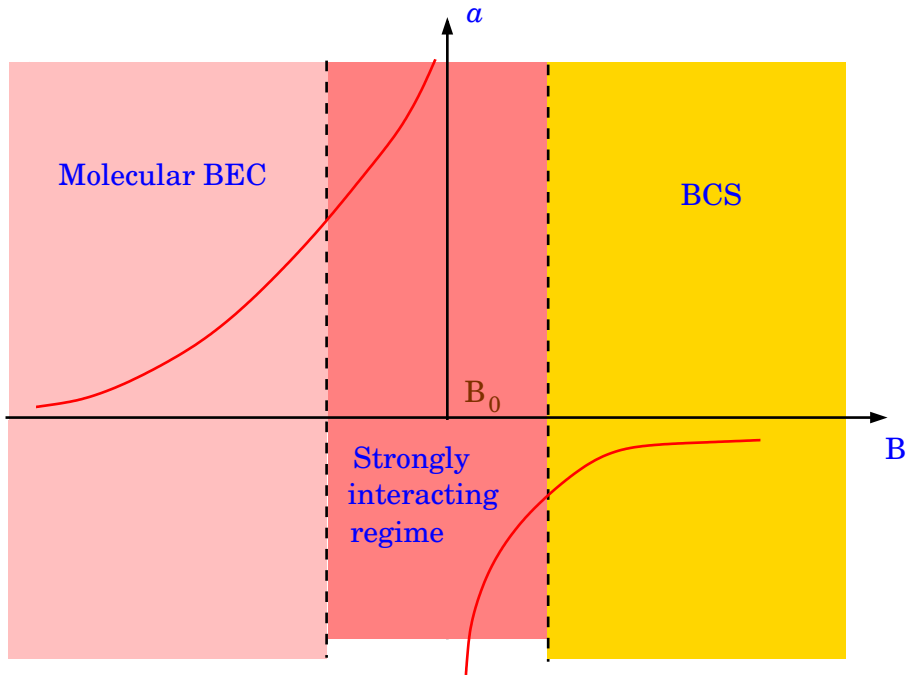


Figure 7.4: Phase diagram for a two-component atomic Fermi gas at $T = 0$.

7.4 Description of the strongly interacting regime. Unitarity limit

We now treat all regimes shown in Fig. 7.4 on equal footing by using a sort of BCS approach at $T = 0$. So, we again write the BCS Hamiltonian (4.23)

$$\hat{H}_{\text{BCS}} = \sum_{\mathbf{k}; \sigma=\uparrow, \downarrow} \xi_{\mathbf{k}} \hat{a}_{\sigma\mathbf{k}}^{\dagger} \hat{a}_{\sigma\mathbf{k}} + \Delta \sum_{\mathbf{k}} (\hat{a}_{\downarrow\mathbf{k}} \hat{a}_{\uparrow-\mathbf{k}} + \hat{a}_{\uparrow-\mathbf{k}}^{\dagger} \hat{a}_{\downarrow\mathbf{k}}^{\dagger}),$$

where $\hat{a}_{\sigma\mathbf{k}}$ are the particle operators, and $\xi_{\mathbf{k}} = (\hbar^2 k^2/2m - \mu)$. We then employ the Bogoliubov transformation (4.24), (4.25):

$$\hat{b}_{\uparrow\mathbf{k}} = u_k \hat{a}_{\uparrow\mathbf{k}} + v_k \hat{a}_{\downarrow-\mathbf{k}}^\dagger,$$

$$\hat{b}_{\downarrow\mathbf{k}} = u_k \hat{a}_{\downarrow\mathbf{k}} - v_k \hat{a}_{\uparrow-\mathbf{k}}^\dagger,$$

thus reducing \hat{H}_{BCS} to the diagonal form (4.31):

$$\hat{H}_{\text{BCS}} = E_0 + \sum_{\sigma,\mathbf{k}} \epsilon_k \hat{b}_{\sigma\mathbf{k}}^\dagger \hat{b}_{\sigma\mathbf{k}}.$$

For the ground state energy we have:

$$E_0 = \sum_{\mathbf{k}} \left(2E_k v_k^2 - \frac{\Delta^2}{2\epsilon_k} \right), \quad (7.13)$$

where $E_k = \hbar^2 k^2/2m$, and the normalization condition reads:

$$n = \frac{2}{V} \sum_{\mathbf{k}} v_k^2 = \int \frac{d^3k}{(2\pi)^3} \left(1 - \frac{\xi_k}{\epsilon_k} \right). \quad (7.14)$$

However, in contrast to the standard BCS approach we now do not assume that only momenta close to k_F are important in the integral over d^3k in Eq. (7.14) and use the exact expression $\xi_k = \hbar^2 k^2/2m - \mu$.

The zero-temperature gap equation (4.40) we now write using the so-called renormalized coupling constant:

$$\frac{1}{|g|} = \int \frac{d^3k}{2(2\pi)^3} \left(\frac{1}{\epsilon_k} - \frac{1}{E_k} \right). \quad (7.15)$$

The renormalization of the coupling constant is needed in order to circumvent the divergence of the integral at large k in Eq. (7.15). In the standard BCS approach in *Lecture 4* we did this "by hands", saying that there should be a natural high-energy cut-off which is of the order of the Fermi energy. The rigorous renormalization procedure is beyond the scope of the present course, and we only give supporting arguments. Imagine that we have two particles with very low momenta and treat g as the Fourier transform of the interaction potential. Then, to first order in perturbation theory the coupling constant is g , and the second order gives the contribution $-\sum_{\mathbf{k}} g^2/2E_k V$. So, we have to make a replacement $g \rightarrow g(1 - \sum_{\mathbf{k}} g/2E_k V)$, or

$$\frac{1}{g} \rightarrow \frac{1}{g} + \sum_{\mathbf{k}} \frac{1}{2E_k V}.$$

For a negative g this is exactly what we have done in Eq. (7.15).

The functions u_k, v_k are still determined by the Bogoliubov-de Gennes equations (4.32), (4.33) and are given by Eqs. (4.34), (4.35):

$$u_k^2 = \frac{1}{2} \left(1 + \frac{\xi_k}{\epsilon_k} \right),$$

$$v_k^2 = \frac{1}{2} \left(1 - \frac{\xi_k}{\epsilon_k} \right)$$

The excitation energy ϵ_k follows from equation (4.36):

$$\epsilon_k = \sqrt{\xi_k^2 + \Delta^2}.$$

Then, using Eqs. (7.14) and (7.15), for small $a < 0$ we recover the earlier obtained result for the gap in the BCS limit. We even find a numerical coefficient in front of the exponent. However, this coefficient has to be corrected by including second order many-body processes.

In the limit of molecular BEC ($a > 0$ and $na^3 \ll 1$) equations (7.14), (7.15) and (7.13) give the ground state energy

$$E_0 = -\frac{\hbar^2}{2ma^2}N + \frac{\pi\hbar^2 a}{2m} \frac{N^2}{V}, \quad (7.16)$$

where N is the number of fermionic atoms. Accordingly, the number of molecules is $N_m = N/2$, and the first term in Eq. (7.16) is nothing else than the binding energy of a molecule, $\epsilon_0 = \hbar^2/ma^2$, multiplied by the number of molecules. The second term corresponds to the interaction between the molecules. Taking into account that the molecular mass is $2m$, we see that the molecule-molecule scattering length is $a_m = 2a$. This is because the molecule-molecule interaction energy should be equal to $(4\pi\hbar^2 a_m/2m)N_m^2/2V$. Note that the exact result following from the solution of the 4-body problem gives $a_m = 0.6a$.

The chemical potential following from Eq. (7.16) is

$$\mu = -\frac{\epsilon_0}{2} + \frac{\pi\hbar^2}{m}an. \quad (7.17)$$

The order parameter Δ then turns out to be much smaller than $|\mu|$, but the gap in the excitation spectrum is provided by the binding energy of a molecule and is equal to $\epsilon_0/2$.

In the unitarity limit, that is at $|a| \rightarrow \infty$, one finds $\Delta = 0.69E_F$, $E_0 = 0.6(3E_F/5)N$, and $\mu = 0.6E_F$. The latter result is 40% larger than the result of the exact Monte Carlo calculation.

We thus see that even using the BCS theory one can qualitatively recover the physics. The discussed approach for describing the BCS-BEC crossover has been proposed by Leggett. Note that the unitarity limit is universal. There is only one energy scale, the Fermi energy E_F , and only one distance scale, the mean interparticle separation $n^{-1/3}$.

Monte Carlo studies have obtained the superfluid transition temperature T_c as a function of the scattering length a , which for a given system is easily transformed into the dependence of T_c on the magnetic field. This dependence is sketched in Fig. 7.5. In the unitarity limit we have $T_c \approx 0.15E_F$. Experimental studies have achieved the strongly interacting regime, and Monte Carlo calculations of T_c and μ are consistent with the experiments. The presence of superfluidity in the strongly interacting regime has been proven in MIT experiments (W. Ketterle) by creating a lattice of quantum vortices.

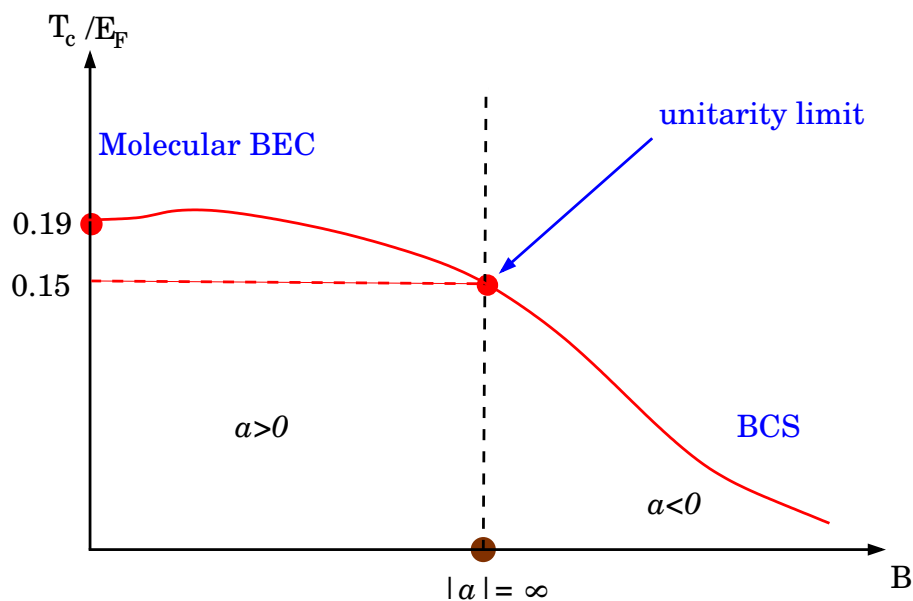


Figure 7.5: Superfluid transition temperature as a function of the magnetic field for the BCS-BEC crossover in a two-component atomic Fermi gas.

Problems 7

7.1 Calculate the spectrum of collective excitations for a 2-component Fermi gas in the unitarity limit and in the regime of molecular BEC.

We rely on the hydrodynamic equations (5.12), (5.13):

$$\frac{\partial n}{\partial t} + \text{div}(n\mathbf{v}) = 0,$$

$$m \frac{\partial \mathbf{v}}{\partial t} + \nabla \left[\frac{1}{2} m v^2 + \mu(n) \right] = 0,$$

and assume small fluctuations of the density, $\delta n = n - \bar{n}$, where \bar{n} is the mean density. Then, writing the velocity as $\mathbf{v} = (\hbar/m)\nabla\phi$, with ϕ being the phase of the order parameter, we arrive at the linearized equations (5.14), (5.15):

$$\begin{aligned}\frac{\partial\delta n}{\partial t} + \nabla\left(\frac{\hbar\bar{n}}{m}\nabla\phi\right) &= 0, \\ \hbar\frac{\partial\nabla\phi}{\partial t} + \frac{\partial\mu}{\partial n}\Big|_{n=\bar{n}}\nabla\delta n &= 0.\end{aligned}$$

Reducing them to Eq. (5.16):

$$\frac{\partial^2\delta n}{\partial t^2} = \frac{\bar{n}}{m}\frac{\partial\mu}{\partial n}\Big|_{n=\bar{n}}\nabla^2\delta n,$$

we then write small fluctuations of the density as $\delta n \propto \exp(-i\omega t + i\mathbf{k}\mathbf{r})$. This gives

$$\omega^2 = \left(\frac{\bar{n}}{m}\frac{\partial\mu}{\partial n}\Big|_{n=\bar{n}}\right)k^2. \quad (7.18)$$

In the unitarity limit we have $\mu = 0.4E_F$ and

$$\frac{\partial\mu}{\partial n}\Big|_{n=\bar{n}} = 0.4 \times \frac{2E_F}{3\bar{n}} = 0.267\frac{E_F}{\bar{n}},$$

which yields

$$\omega = \left(\frac{0.4}{3}\right)^{1/2}v_F k = 0.365v_F k. \quad (7.19)$$

In the regime of molecular BEC we have

$$\mu = -\frac{\epsilon_0}{2} + \frac{\pi\hbar^2}{2m}n \times 0.6a$$

and

$$\frac{\partial\mu}{\partial n}\Big|_{n=\bar{n}} = 0.3\frac{\pi\hbar^2 a}{m}.$$

Then equation (7.18) yields

$$\omega = \left(0.3\frac{\pi\hbar^2\bar{n}a}{m^2}\right)^{1/2}k \simeq 0.314(\bar{n}a^3)^{1/6}v_F k. \quad (7.20)$$

7.2 Calculate the lowest frequencies of collective excitations of a 2-component Fermi gas in the unitarity regime in a harmonic potential $V(r) = m\omega_0^2 r^2/2$.